

# LOCALIC COMPLETION OF UNIFORM SPACES

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**ABSTRACT.** We extend the notion of localic completion of generalised metric spaces by Steven Vickers to the setting of generalised uniform spaces. A generalised uniform space (gus) is a set  $X$  equipped with a family of generalised metrics on  $X$ , where a generalised metric on  $X$  is a map from  $X \times X$  to the upper reals satisfying zero self-distance law and triangle inequality.

For a symmetric generalised uniform space, the localic completion lifts its generalised uniform structure to a point-free generalised uniform structure. This point-free structure induces a complete generalised uniform structure on the set of formal points of the localic completion that gives the standard completion of the original gus with Cauchy filters.

We extend the localic completion to a full and faithful functor from the category of locally compact uniform spaces into that of overt locally compact completely regular formal topologies. Moreover, we give an elementary characterisation of the cover of the localic completion of a locally compact uniform space that simplifies the existing characterisation for metric spaces. These results generalise the corresponding results for metric spaces by Erik Palmgren.

Furthermore, we show that the localic completion of a symmetric gus is equivalent to the point-free completion of the uniform formal topology associated with the gus.

We work in Aczel's constructive set theory CZF with the Regular Extension Axiom. Some of our results also require Countable Choice.

## 1. INTRODUCTION

Formal topology [Sam87] is a predicative presentation of locales, and has been successful in constructivising many results of the classical topology. However, the relation between formal topology, which is a point-free notion, and other constructive point-set approaches to topology is not as simple as in the classical case.

Classically, the adjunction between the category of topological spaces and that of locales provides a fundamental tool which relates locale theory and point-set topology. Although this adjunction has been shown to be constructively valid by Aczel [Acz06, Theorem 21], it

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seems to be of little practical use in Bishop constructive mathematics [Bis67] since we cannot obtain useful correspondence between point-set notions and point-free notions through this adjunction. For example, we cannot show that the formal reals and formal Cantor space are equivalent to their point-set counterparts without employing Fan theorem [FG82, Theorem 3.4] [GS07, Proposition 4.3], which is not valid in the recursive realizability interpretation (see [Tv88, Chapter 4, Section 7]). Since Bishop emphasises the computational aspect of his mathematics [Bis67, Appendix B], it seems that Fan theorem is not acceptable in Bishop constructive mathematics, the development of which we respect and follow. Moreover, from the topos theoretic point of view [MLM92], Fan theorem is not desirable as it is not valid in some Grothendieck toposes [FH79, Section 4].

Instead, Palmgren used Vickers's notion of localic completion of generalised metric spaces [Vic05a] to construct another embedding from the category of locally compact metric spaces into that of formal topologies that has more desirable properties, e.g. preservation of compactness and local compactness, and the order of real valued continuous functions [Pal07]. This allows us to transfer results in Bishop's theory of metric spaces to formal topologies.

The main aim of this paper is to further strengthen the connection between Bishop constructive mathematics and formal topology by generalising the results by Vickers and Palmgren to the setting of uniform spaces. The notion of uniform space that we deal with in this paper is a set equipped with a family of generalised metrics on it, where a generalised metric is the notion obtained by dropping the symmetry from that of pseudometric and allowing it to take values in the non-finite upper reals (Definition 3.1). This structure will be called a generalised uniform space (gus). When specialised to a family of pseudometrics (i.e. finite Dedekind symmetric generalised metrics), the notion corresponds to that of uniform space treated in the book by Bishop [Bis67, Chapter 4, Problems 17].<sup>1</sup> Hence, the notion of generalised uniform space provides a natural setting to talk about generalisations of the results by Vickers and Palmgren at the same time. Our first aim is to extend the notion of localic completion of generalised metric spaces by Vickers to gus's (Section 3). Our second aim is to extend Palmgren's results about Bishop metric spaces [Pal07] to Bishop uniform spaces by applying the localic completion of gus's (Section 4). Our third aim is to relate the localic completion to the point-free completion of uniform formal topologies by Fox [Fox05, Chapter 6] (Section 5).

We now summarise our main results. In Section 3, we define the localic completion of a generalised uniform space, and examine its functorial properties and uniform structure. In particular, we show that the localic completion preserves inhabited countable products of gus's (Theorem 3.15). For a symmetric gus  $X$  with a family of generalised metrics  $M$ , we show that the localic completion of  $X$  is embedded into the point-free product of the localic completion of each generalised metric in  $M$  as an overt weakly closed subtopology (Theorem 3.19). This is a point-free analogue of the usual point-set completion of  $X$ . Next, we show that the formal points of the localic completion of  $X$  gives the standard completion of  $X$  in terms of Cauchy filters (Theorem 3.30). In Section 4, we specialise the localic completion to the class of finite Dedekind symmetric gus's, and extend Palmgren's functorial embedding of locally compact metric spaces to uniform spaces (Theorem 4.23). Here, one of our important contributions is a new characterisation of the cover of the localic completion of a locally compact uniform space (Proposition 4.9) that simplifies the previous

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<sup>1</sup>Other well-known notions of uniform spaces are a set equipped with an entourage uniformity and a set equipped with a covering uniformity.

characterisation for locally compact metric spaces by Palmgren [Pal07, Theorem 4.17]. In Section 5, we show that the localic completion of a symmetric gus  $X$  is equivalent to the point-free completion of the uniform formal topology associated with  $X$  (Theorem 5.5).

We work informally in Aczel's constructive set theory CZF extended with the Regular Extension Axiom (REA) [AR01, Section 5]. REA is needed to define the notion of inductively generated formal topology (see Section 2.1). Some of the results in Section 3.4.3 also require Countable Choice ( $\text{AC}_\omega$ ) (see Remark 3.31). We assume Palmgren's work [Pal07] which is carried out in Bishop's naïve set theory [Bis67] with some generalised inductive definitions. It is our understanding that his work can be carried out in  $\text{CZF} + \text{REA} + \text{AC}_\omega$ , but we take extra care not to let our results depend on  $\text{AC}_\omega$  implicitly.

**Notation 1.1.** We fix some notations. For any set  $S$ ,  $\text{Pow}(S)$  denotes the class of subsets of  $S$ . Note that since CZF is predicative,  $\text{Pow}(S)$  cannot be shown to be a set unless  $S = \emptyset$ .  $\text{Fin}(S)$  denotes the *set* of finitely enumerable subsets of  $S$ , where a set  $A$  is finitely enumerable if there exists a surjection  $f: \{0, \dots, n-1\} \rightarrow A$  for some  $n \in \mathbb{N}$ .  $\text{Fin}^+(S)$  denotes the set of inhabited finitely enumerable subsets of  $S$ . For subsets  $U, V \subseteq S$ , we define

$$U \wp V \stackrel{\text{def}}{\iff} (\exists a \in S) a \in U \cap V.$$

Given a relation  $r \subseteq S \times T$  between sets  $S$  and  $T$  and their subsets  $U \subseteq S$  and  $D \subseteq T$ , we define

$$\begin{aligned} rU &\stackrel{\text{def}}{=} \{b \in T \mid (\exists a \in U) a \, r \, b\}, \\ r^-D &\stackrel{\text{def}}{=} \{a \in S \mid (\exists b \in D) a \, r \, b\}, \\ r^{-*}U &\stackrel{\text{def}}{=} \{b \in T \mid r^- \{b\} \subseteq U\}. \end{aligned}$$

We often write  $r^-b$  for  $r^- \{b\}$ . The set theoretic complement of a subset  $U \subseteq S$  is denoted by  $\neg U$ , i.e.  $\neg U \stackrel{\text{def}}{=} \{a \in S \mid \neg(a \in U)\}$ .

## 2. FORMAL TOPOLOGIES

This section provides background on formal topologies to our main results in subsequent sections. Our main reference of formal topologies is [Fox05], where overt formal topologies are called open formal topologies. The exposition of the point-free real numbers in Section 2.4.3 follows that of Vickers [Vic05a]. Nothing in section is essentially new; hence a knowledgeable reader is advised to skip this section.

It should be noted that many results on formal topologies are originally obtained in Martin-Löf's type theory. So far, not all of them are available in  $\text{CZF} + \text{REA} (+\text{AC}_\omega)$  (see [vdB13, Introduction]); however, our results do not depend on those results which are only available in type theory. In any case, we will provide references to the corresponding results in CZF in case the original results were obtained in type theory.

**Definition 2.1.** A *formal topology*  $\mathcal{S}$  is a triple  $(S, \triangleleft, \leq)$  where  $(S, \leq)$  is a preordered set and  $\triangleleft$  is a relation between  $S$  and  $\text{Pow}(S)$  such that

$$\mathcal{A}U \stackrel{\text{def}}{=} \{a \in S \mid a \triangleleft U\}$$

is a set for each  $U \subseteq S$  and that

- (1)  $U \triangleleft U$ ,
- (2)  $a \triangleleft U \ \& \ U \triangleleft V \implies a \triangleleft V$ ,
- (3)  $a \triangleleft U \ \& \ a \triangleleft V \implies a \triangleleft U \downarrow V$ ,
- (4)  $a \leq b \implies a \triangleleft b$

for all  $a, b \in S$  and  $U, V \subseteq S$ , where

$$U \triangleleft V \stackrel{\text{def}}{\iff} (\forall a \in U) a \triangleleft V,$$

$$U \downarrow V \stackrel{\text{def}}{=} \{c \in S \mid (\exists a \in U) (\exists b \in V) c \leq a \ \& \ c \leq b\}.$$

We write  $a \downarrow U$  for  $\{a\} \downarrow U$  and  $U \triangleleft a$  for  $U \triangleleft \{a\}$ . The set  $S$  is called the *base* of  $\mathcal{S}$ , and the relation  $\triangleleft$  is called a *cover* on  $S$ .

The class  $\text{Sat}(\mathcal{S}) = \{\mathcal{A}U \mid U \in \mathbf{Pow}(S)\}$  forms a frame with the top  $S$ , the meet  $\mathcal{A}U \wedge \mathcal{A}V = \mathcal{A}(U \downarrow V)$  and the join  $\bigvee_{i \in I} \mathcal{A}U_i = \mathcal{A}\bigcup_{i \in I} U_i$  for each  $U, V \subseteq S$  and for each set-indexed family  $(U_i)_{i \in I}$  of subsets of  $S$ .

Formal topologies will be denoted by  $\mathcal{S}, \mathcal{S}', \dots$ , and their underlying bases and covers will be denoted by  $S, S', \dots$ , and  $\triangleleft, \triangleleft', \dots$  respectively.

**Definition 2.2.** Let  $\mathcal{S}$  and  $\mathcal{S}'$  be formal topologies. A relation  $r \subseteq S \times S'$  is called a *formal topology map* from  $\mathcal{S}$  to  $\mathcal{S}'$  if

- (FTM1)  $S \triangleleft r^- S'$ ,
- (FTM2)  $r^- \{a\} \downarrow r^- \{b\} \triangleleft r^- (a \downarrow' b)$ ,
- (FTM3)  $a \triangleleft' U \implies r^- a \triangleleft r^- U$

for all  $a, b \in S'$  and  $U \subseteq S'$ . The class  $\text{Hom}(\mathcal{S}, \mathcal{S}')$  of formal topology maps from  $\mathcal{S}$  to  $\mathcal{S}'$  is ordered by

$$r \leq s \stackrel{\text{def}}{\iff} (\forall a \in S') r^- a \triangleleft s^- a.$$

Two formal topology maps  $r, s : \mathcal{S} \rightarrow \mathcal{S}'$  are defined to be *equal*, denoted by  $r = s$ , if  $r \leq s$  and  $s \leq r$ .

A formal topology map  $r : \mathcal{S} \rightarrow \mathcal{S}'$  bijectively corresponds to a frame homomorphism  $F_r : \text{Sat}(\mathcal{S}') \rightarrow \text{Sat}(\mathcal{S})$  between the associated frames in such a way that  $F_r(\mathcal{A}'U) = \mathcal{A}r^-U$ .

The formal topologies and formal topology maps between them form a category **FTop**. The composition of two formal topology maps is the composition of the underlying relations of these maps. The identity morphism on a formal topology is the identity relation on its base.

The formal topology  $\mathbf{1} \stackrel{\text{def}}{=} (\{*\}, \in, =)$  is a terminal object in **FTop**. A formal point of a formal topology  $\mathcal{S}$  is a formal topology map  $r : \mathbf{1} \rightarrow \mathcal{S}$ . An equivalent definition is the following.

**Definition 2.3.** Let  $\mathcal{S}$  be a formal topology. A subset  $\alpha \subseteq S$  is a *formal point* of  $\mathcal{S}$  if

- (P1)  $S \not\triangleleft \alpha$ ,
- (P2)  $a, b \in \alpha \implies \alpha \not\triangleleft (a \downarrow b)$ ,
- (P3)  $a \in \alpha \ \& \ a \triangleleft U \implies \alpha \not\triangleleft U$

for all  $a, b \in S$  and  $U \subseteq S$ . The class of formal points of  $\mathcal{S}$  is denoted by  $\text{Pt}(\mathcal{S})$ . Predicatively, we cannot assume that  $\text{Pt}(\mathcal{S})$  is a set.<sup>2</sup>

<sup>2</sup>For example, for any set  $S$ , the class of formal points of a formal topology  $(\text{Fin}(S), \triangleleft, \leq)$ , where  $A \leq B \stackrel{\text{def}}{\iff} B \subseteq A$  and  $A \triangleleft U \stackrel{\text{def}}{\iff} (\exists B \in \mathcal{U}) A \leq B$ , is in one-one correspondence with  $\mathbf{Pow}(S)$ .

Impredicatively, the formal points  $\text{Pt}(\mathcal{S})$  form a topological space with the topology generated by the basic opens of the form

$$a_* \stackrel{\text{def}}{=} \{\alpha \in \text{Pt}(\mathcal{S}) \mid a \in \alpha\}$$

for each  $a \in S$ . If  $r: \mathcal{S} \rightarrow \mathcal{S}'$  is a formal topology map, then the function  $\text{Pt}(r): \text{Pt}(\mathcal{S}) \rightarrow \text{Pt}(\mathcal{S}')$  given by

$$\text{Pt}(r)(\alpha) \stackrel{\text{def}}{=} r\alpha$$

for each  $\alpha \in \text{Pt}(\mathcal{S})$  is a well defined continuous function with respect to the topologies on  $\text{Pt}(\mathcal{S})$  and  $\text{Pt}(\mathcal{S}')$ . Impredicatively, the operation  $\text{Pt}(-)$  is the right adjoint of the adjunction between the category of topological spaces and formal topologies established by Aczel [Acz06].

From Section 3 onward, we mainly work with overt formal topologies, i.e. formal topologies equipped with a positivity predicate.

**Definition 2.4.** Let  $\mathcal{S}$  be a formal topology. A subset  $V \subseteq S$  is said to be *splitting* if

$$a \in V \ \& \ a \triangleleft U \implies V \not\Downarrow U$$

for all  $a \in S$  and  $U \subseteq S$ . A *positivity predicate* (or just a positivity) on a formal topology  $\mathcal{S}$  is a splitting subset  $\text{Pos} \subseteq S$  that satisfies

$$a \triangleleft \{x \in S \mid x = a \ \& \ \text{Pos}(a)\} \tag{Pos}$$

for all  $a \in S$ , where we write  $\text{Pos}(a)$  for  $a \in \text{Pos}$ . A formal topology is *overt* if it is equipped with a positivity predicate.

Let  $\mathcal{S}$  be a formal topology. By condition (Pos), a positivity predicate on  $\mathcal{S}$ , if it exists, is the largest splitting subset of  $\mathcal{S}$ . Thus, a formal topology admits at most one positivity predicate.

**2.1. Inductively generated formal topologies.** The notion of inductively generated formal topology by Coquand et al. [CSSV03] allows us to define a formal topology by a small set of axioms.

**Definition 2.5.** Let  $S$  be a set. An *axiom-set* on  $S$  is a pair  $(I, C)$ , where  $(I(a))_{a \in S}$  is a family of sets indexed by  $S$ , and  $C$  is a family  $(C(a, i))_{a \in S, i \in I(a)}$  of subsets of  $S$  indexed by  $\sum_{a \in S} I(a)$ .

The following result, which was obtained in Martin-Löf's type theory, is also valid in CZF + REA [Acz06, Section 6].

**Theorem 2.6** ([CSSV03, Theorem 3.3]). *Let  $(S, \leq)$  be a preordered set, and let  $(I, C)$  be an axiom-set on  $S$ . Then, there exists a cover  $\triangleleft_{I, C}$  inductively generated by the following rules:*

$$\begin{array}{c} \frac{a \in U}{a \triangleleft_{I, C} U} \text{ (reflexivity); } \quad \frac{a \leq b \quad b \triangleleft_{I, C} U}{a \triangleleft_{I, C} U} \text{ } (\leq\text{-left}); \\ \frac{a \leq b \quad i \in I(b) \quad \{a\} \downarrow C(b, i) \triangleleft_{I, C} U}{a \triangleleft_{I, C} U} \text{ } (\leq\text{-infinity}). \end{array}$$

The relation  $\triangleleft_{I, C}$  is the least cover on  $S$  which satisfies  $(\leq\text{-left})$  and  $a \triangleleft_{I, C} C(a, i)$  for each  $a \in S$  and  $i \in I(a)$ , and  $\triangleleft_{I, C}$  is called the cover inductively generated by  $(I, C)$ .

A formal topology  $\mathcal{S} = (S, \triangleleft, \leq)$  is *inductively generated* if it is equipped with an axiom-set  $(I, C)$  on  $S$  such that  $\triangleleft = \triangleleft_{I, C}$ .

Localised axiom-sets are particularly convenient to work with.

**Definition 2.7.** Let  $(S, \leq)$  be a preordered set, and let  $(I, C)$  be an axiom-set on  $S$ . We say that  $(I, C)$  is *localised* with respect to  $(S, \leq)$  if

$$a \leq b \implies (\forall i \in I(b)) (\exists j \in I(a)) C(a, j) \subseteq a \downarrow C(b, i)$$

for all  $a, b \in S$ . We often leave implicit the preorder with respect to which the axiom-set is localised. In that case, the context will always make it clear what is left implicit.

For an axiom-set  $(I, C)$  which is localised with respect to a preorder  $(S, \leq)$ , we can replace ( $\leq$ -infinity) rule in Theorem 2.6 with (infinity):

$$\frac{i \in I(a) \quad C(a, i) \triangleleft_{I, C} U}{a \triangleleft_{I, C} U} \text{ (infinity)}.$$

Thus, if  $\mathcal{S} = (S, \triangleleft, \leq)$  is the formal topology inductively generated by a localised axiom-set  $(I, C)$ , then for each  $U \subseteq S$ , the set

$$\mathcal{A}U \stackrel{\text{def}}{=} \{a \in S \mid a \triangleleft U\}$$

is the least subset of  $S$  such that

- (1)  $U \subseteq \mathcal{A}U$ ,
- (2)  $a \leq b \ \& \ b \in \mathcal{A}U \implies a \in \mathcal{A}U$ ,
- (3)  $C(a, i) \subseteq \mathcal{A}U \implies a \in \mathcal{A}U$

for all  $a, b \in S$  and  $i \in I(a)$ .

Therefore, we have the following induction principle: let  $(I, C)$  be a localised axiom-set with respect to a preorder  $(S, \leq)$ . Then, for any subset  $U \subseteq S$  and a predicate  $\Phi$  on  $S$ , if

- (ID1)  $\frac{a \in U}{\Phi(a)}$ ,
- (ID2)  $\frac{a \leq b \quad \Phi(b)}{\Phi(a)}$ ,
- (ID3)  $\frac{i \in I(a) \quad (\forall c \in C(a, i)) \Phi(c)}{\Phi(a)}$

for all  $a, b \in S$ , then  $a \triangleleft_{I, C} U \implies \Phi(a)$  for all  $a \in S$ . An application of the above principle is called a *proof by induction on the cover*  $\triangleleft_{I, C}$ .

**Remark 2.8.** In Definition 2.2 of formal topology map, if the formal topology  $\mathcal{S}'$  is inductively generated by an axiom-set  $(I, C)$  on  $S'$ , then the condition (FTM3) is equivalent to the following conditions under the condition (FTM2).

- (FTM3a)  $a \leq' b \implies r^-a \triangleleft r^-b$ ,
- (FTM3b)  $r^-a \triangleleft r^-C(a, i)$

for all  $a, b \in S'$  and  $i \in I(a)$ .

Similarly, in Definition 2.3 of formal point, if the formal topology  $\mathcal{S}$  is inductively generated by an axiom-set  $(I, C)$  on  $S$ , then the condition (P3) is equivalent to the following conditions:

- (P3a)  $a \leq b \ \& \ a \in \alpha \implies b \in \alpha$ ,
- (P3b)  $a \in \alpha \implies \alpha \not\propto C(a, i)$

for all  $a, b \in S$  and  $i \in I(a)$ . See Fox [Fox05, Section 4.1.2] for further details.

In the same setting as above, a subset  $V \subseteq S$  is splitting if and only if the following two conditions hold:

- (Spl1)  $a \in V \ \& \ a \leq b \implies b \in V$ ,
- (Spl2)  $a \leq b \ \& \ a \in V \ \& \ i \in I(b) \implies V \checkmark (a \downarrow C(b, i))$ .

Furthermore, if the axiom-set  $(I, C)$  is localised, the condition (Spl2) can be replaced by

- (Spl2')  $a \in V \ \& \ i \in I(a) \implies V \checkmark C(a, i)$ .

**2.2. Products and pullbacks.** Inductively generated formal topologies are closed under arbitrary limits. We recall the constructions of products and pullbacks.

**2.2.1. Products.** Following Vickers [Vic05b], we define a product of a set-indexed family of inductively generated formal topologies as follows. Let  $(\mathcal{S}_i)_{i \in I}$  be a family of inductively generated formal topologies, each member of which is of the form  $\mathcal{S}_i = (S_i, \triangleleft_i, \leq_i)$ , and let  $(K_i, C_i)$  be the axiom-set which generates  $\mathcal{S}_i$ . Define a preorder  $(S_\Pi, \leq_\Pi)$  by

$$S_\Pi \stackrel{\text{def}}{=} \text{Fin} \left( \sum_{i \in I} S_i \right),$$

$$A \leq_\Pi B \stackrel{\text{def}}{\iff} (\forall (i, b) \in B) (\exists (j, a) \in A) i = j \ \& \ a \leq_i b$$

for all  $A, B \in S_\Pi$ . The axiom-set on  $(S_\Pi, \leq_\Pi)$  is given by

- (S1)  $S_\Pi \triangleleft_\Pi \{ \{(i, a)\} \in S_\Pi \mid a \in S_i \}$  for each  $i \in I$ ,
- (S2)  $\{(i, a), (i, b)\} \triangleleft_\Pi \{ \{(i, c)\} \in S_\Pi \mid c \leq_i a \ \& \ c \leq_i b \}$  for each  $i \in I$  and  $a, b \in S_i$ ,
- (S3)  $\{(i, a)\} \triangleleft_\Pi \{ \{(i, b)\} \in S_\Pi \mid b \in C_i(a, k) \}$  for each  $i \in I$ ,  $a \in S_i$ , and  $k \in K_i(a)$ .

Let  $\prod_{i \in I} \mathcal{S}_i = (S_\Pi, \triangleleft_\Pi, \leq_\Pi)$  be the formal topology inductively generated by the above axiom-set. For each  $i \in I$ , the projection  $p_i: \prod_{i \in I} \mathcal{S}_i \rightarrow \mathcal{S}_i$  is defined by

$$A p_i a \stackrel{\text{def}}{\iff} A = \{(i, a)\}$$

for all  $A \in S_\Pi$  and  $a \in S_i$ .

Given any family  $(r_i: \mathcal{S} \rightarrow \mathcal{S}_i)_{i \in I}$  of formal topology maps, we have a unique formal topology map  $r: \mathcal{S} \rightarrow \prod_{i \in I} \mathcal{S}_i$  such that  $r_i = p_i \circ r$  for each  $i \in I$ . The map  $r$  is defined by

$$a r A \stackrel{\text{def}}{\iff} (\forall (i, b) \in A) a \triangleleft r_i^- b$$

for each  $a \in S$  and  $A \in S_\Pi$ .

**2.2.2. Binary products.** A binary product of a pair of inductively generated formal topologies admits a simple construction. Given two inductively generated formal topologies  $\mathcal{S} = (S, \triangleleft_S, \leq_S)$  and  $\mathcal{T} = (T, \triangleleft_T, \leq_T)$  generated by axiom-sets  $(I, C)$  and  $(J, D)$  respectively, their product  $\mathcal{S} \times \mathcal{T}$  is an inductively generated formal topology with the preorder  $(S \times T, \leq)$  defined by

$$(a, b) \leq (a', b') \stackrel{\text{def}}{\iff} a \leq_S a' \ \& \ b \leq_T b'$$

and the axiom-set  $(K, E)$  on  $(S \times T, \leq)$  defined by

$$\begin{aligned} K((a, b)) &\stackrel{\text{def}}{=} I(a) + J(b), \\ E((a, b), (0, i)) &\stackrel{\text{def}}{=} C(a, i) \times \{b\}, \\ E((a, b), (1, j)) &\stackrel{\text{def}}{=} \{a\} \times D(b, j). \end{aligned}$$

The projection  $p_S: S \times T \rightarrow S$  is given by

$$(a, b) p_S a' \stackrel{\text{def}}{\iff} (a, b) \triangleleft_{K, E} \{a'\} \times T$$

for each  $a, a' \in S$  and  $b \in T$ , and the other projection is similarly defined.

Given two formal topology maps  $r: S' \rightarrow S$  and  $s: S' \rightarrow T$ , the canonical map  $\langle r, s \rangle: S' \rightarrow S \times T$  is given by

$$c \langle r, s \rangle (a, b) \stackrel{\text{def}}{\iff} c \triangleleft' r^{-}a \ \& \ c \triangleleft' s^{-}b$$

for each  $c \in S'$ ,  $a \in S$ , and  $b \in T$ .

**2.2.3. Pullbacks.** Given inductively generated formal topologies  $S_1$  and  $S_2$  generated by axiom-sets  $(I_1, C_1)$  and  $(I_2, C_2)$  respectively, a pullback  $S_1 \times_{\mathcal{T}} S_2$  of formal topology maps  $r: S_1 \rightarrow T$ ,  $s: S_2 \rightarrow T$  is generated by the axiom-set of the product  $S_1 \times S_2$  together with the following additional axioms:

$$\begin{aligned} (a, b) \triangleleft S_1 \times s^{-}c &\quad (a \ r \ c \ \& \ c \in T), \\ (a, b) \triangleleft r^{-}c \times S_2 &\quad (b \ s \ c \ \& \ c \in T). \end{aligned}$$

The restrictions of the projections  $p_i: S_1 \times S_2 \rightarrow S_i$  ( $i = 1, 2$ ) to  $S_1 \times_{\mathcal{T}} S_2$  form a pullback square.

### 2.3. Subtopologies.

**Definition 2.9.** A *subtopology* of a formal topology  $S$  is a formal topology  $S' = (S, \triangleleft', \leq)$  where  $\triangleleft'$  is a cover on  $S$  and  $(S, \leq)$  is the underlying preorder of  $S$  such that  $\mathcal{A}U \subseteq \mathcal{A}'U$  for each  $U \subseteq S$ . If  $S'$  is a subtopology of  $S$ , we write  $S' \sqsubseteq S$ .

Given a formal topology map  $r: S \rightarrow S'$ , the relation  $\triangleleft_r \subseteq S' \times \text{Pow}(S')$  given by

$$a \triangleleft_r U \stackrel{\text{def}}{\iff} a \in r^{-*} \mathcal{A} r^{-}U$$

is a cover on  $S'$ . The formal topology  $S_r = (S', \triangleleft_r, \leq')$  is called the *image* of  $S$  under  $r$ . A formal topology map  $r: S \rightarrow S'$  is an *embedding* if  $r$  restricts to an isomorphism between  $S$  and its image  $S_r$ , and  $r$  is called a *surjection* if its image is  $S'$ . It can be shown that  $r: S \rightarrow S'$  is an embedding if and only if

$$a \triangleleft r^{-}r^{-*} \mathcal{A} \{a\}$$

for all  $a \in S$ . See Fox [Fox05, Proposition 3.5.2].

By the condition (FTM3) for a formal topology map, we have  $S_r \sqsubseteq S'$  for any formal topology map  $r: S \rightarrow S'$ . If  $S'$  is a subtopology of  $S = (S, \triangleleft, \leq)$ , then the identity relation  $\text{id}_S$  on  $S$  is an embedding  $\text{id}_S: S' \rightarrow S$ . Hence the notion of embedding is essentially equivalent to that of subtopology.

The following is well known. We omit an easy proof.



**Lemma 2.10.** *Let  $\mathcal{S}$  be an overt formal topology with a positivity  $\text{Pos}$ , and let  $r: \mathcal{S} \rightarrow \mathcal{S}'$  be a formal topology map. Then, the image  $\mathcal{S}_r$  of  $\mathcal{S}$  under  $r$  is overt with the positivity*

$$r \text{ Pos} = \{a \in \mathcal{S}' \mid (\exists b \in \text{Pos}) b \text{ } r \text{ } a\}.$$

The notion of weakly closed subtopology due to Bunge and Funk [BF96] is particularly relevant to us (see Vickers [Vic07] and Fox [Fox05] for treatments in formal topology). For the reason of predicativity, we only consider overt weakly closed subtopologies of inductively generated formal topologies.

**Definition 2.11.** Let  $\mathcal{S}$  be an inductively generated formal topology, and let  $V \subseteq \mathcal{S}$  be a splitting subset of  $\mathcal{S}$ . The overt *weakly closed subtopology* of  $\mathcal{S}$  determined by  $V$ , denoted by  $\mathcal{S}_V$ , is inductively generated by the axioms of  $\mathcal{S}$  together with the following extra axioms:

$$a \triangleleft_V V \cap \{a\}$$

for each  $a \in \mathcal{S}$ . In this case,  $\mathcal{S}_V$  is overt with positivity  $V$ .

Clearly,  $\mathcal{S}_V$  is the largest subtopology of  $\mathcal{S}$  with positivity  $V$ . Moreover, we have  $\text{Pt}(\mathcal{S}_V) = \{\alpha \in \text{Pt}(\mathcal{S}) \mid \alpha \subseteq V\}$  by Remark 2.8, and  $\text{Pt}(\mathcal{S}_V)$  forms a closed subspace of  $\text{Pt}(\mathcal{S})$ .

**2.4. Upper reals and Dedekind reals.** We introduce the notions of upper reals and (possibly non-finite) Dedekind reals, which we use for the values of distance functions of metric spaces. These real numbers are defined as models of propositional geometric theories.

**2.4.1. Propositional geometric theories.** Along with the notion of inductively generated formal topology, propositional geometric theories (i.e. presentations of frames by generators and relations [Vic89, Chapter 2]) provide us with equivalent, but more logical descriptions of formal topologies. Vickers [Vic06] gives a good exposition of the connection between frame presentations and inductively generated formal topologies (see also [Fox05, Chapter 4]).

**Definition 2.12.** Let  $G$  be a set, whose elements are called *propositional symbols* or *generators*. A *propositional geometric theory*  $T$  over  $G$  is a set  $R_T \subseteq \text{Fin}(G) \times \text{Pow}(\text{Fin}(G))$  of *axioms*. An axiom  $(P, \{P_i \mid i \in I\}) \in R_T$  is usually denoted by

$$\bigwedge P \vdash \bigvee_{i \in I} \bigwedge P_i$$

or

$$p_0 \wedge \cdots \wedge p_{n-1} \vdash \bigvee_{i \in I} p_0^i \wedge \cdots \wedge p_{n_i-1}^i.$$

We write  $\top$  for  $\bigwedge \emptyset$ . Henceforth, propositional geometric theories will be simply called geometric theories. A geometric theory over propositional symbols  $G$  with axioms  $R$  will be denoted by a pair  $(G, R)$ .

Every geometric theory  $T = (G, R)$  determines a formal topology  $\mathcal{S}_T = (S_T, \triangleleft_T, \leq_T)$  in the following way. The base  $S_T$  is  $\text{Fin}(G)$  ordered by  $A \leq_T B \iff B \subseteq A$ . The cover  $\triangleleft_T$  is generated by the following axioms:

$$P \triangleleft_T \{P_i \mid i \in I\}$$

for each  $\bigwedge P \vdash \bigvee_{i \in I} \bigwedge P_i \in R$ . The topology  $\mathcal{S}_T$  represents a frame *presented* by the theory in the following sense:

- (1) there exists a function  $\iota_T: G \rightarrow \text{Sat}(\mathcal{S}_T)$ , defined by  $\iota_T(p) = \mathcal{A}_T\{p\}$ , which preserves all the axioms in  $R$ , i.e.

$$\iota_T(p_0) \downarrow \cdots \downarrow \iota_T(p_{n-1}) \triangleleft_T \bigcup_{i \in I} \iota_T(p_0^i) \downarrow \cdots \downarrow \iota_T(p_{n_i-1}^i)$$

for each  $p_0 \wedge \cdots \wedge p_{n-1} \vdash \bigvee_{i \in I} p_0^i \wedge \cdots \wedge p_{n_i-1}^i \in R$ .

- (2) for any frame  $X$  and a function  $f: G \rightarrow Y$  which preserves all the axioms in  $R$ , there exists a unique frame homomorphism  $F: \text{Sat}(\mathcal{S}_T) \rightarrow X$  such that  $F \circ \iota_T = f$ , where  $F$  is given by

$$F(\mathcal{A}U) \stackrel{\text{def}}{=} \bigvee_{P \in U} \bigwedge_{p \in P} f(p)$$

for each  $U \subseteq \mathcal{S}_T$ .

In particular, each formal topology map  $r: \mathcal{S} \rightarrow \mathcal{S}_T$  is determined by a function  $f: G \rightarrow \text{Pow}(S)$  such that

$$f(p_0) \downarrow \cdots \downarrow f(p_{n-1}) \triangleleft \bigcup_{i \in I} f(p_0^i) \downarrow \cdots \downarrow f(p_{n_i-1}^i)$$

for each  $p_0 \wedge \cdots \wedge p_{n-1} \vdash \bigvee_{i \in I} p_0^i \wedge \cdots \wedge p_{n_i-1}^i \in R$ . In this case, we have

$$a \triangleleft r^- \{P\} \iff (\forall p \in P) a \triangleleft f(p)$$

for each  $P \in \text{Fin}(G)$ .

**Remark 2.13.** Adding axioms to a geometric theory  $T$  amounts to defining a subtopology  $\mathcal{S}'$  of  $\mathcal{S}_T$ . Indeed, the identity function on the propositional symbols  $G$  of  $T$  gives rise to the canonical subspace inclusion from  $\mathcal{S}'$  to  $\mathcal{S}_T$  represented by the identity relation on  $\text{Fin}(G)$ .

**2.4.2. Models of theories.** A *model* of a geometric theory  $T = (G, R)$  is a subset  $m \subseteq G$  such that

$$P \subseteq m \implies (\exists i \in I) P_i \subseteq m$$

for each axiom  $\bigwedge P \vdash \bigvee_{i \in I} \bigwedge P_i \in R$ . Let  $\text{Mod}(T)$  denote the class of models of  $T$ . There exists a bijective correspondence between the models of  $T$  and the formal points of  $\mathcal{S}_T$ :

$$\begin{aligned} m &\mapsto \text{Fin}(m): \text{Mod}(T) \rightarrow \text{Pt}(\mathcal{S}_T), \\ \alpha &\mapsto \{p \in G \mid \{p\} \in m\}: \text{Pt}(\mathcal{S}_T) \rightarrow \text{Mod}(T). \end{aligned}$$

The class  $\text{Mod}(T)$  of models gives rise to a topological space, the topology of which is generated by the subbasics of the form

$$a_* \stackrel{\text{def}}{=} \{m \in \text{Mod}(T) \mid a \in m\}.$$

The above bijective correspondence between  $\text{Mod}(T)$  and  $\text{Pt}(\mathcal{S}_T)$  determines homeomorphisms between the associated spaces of  $\text{Mod}(T)$  and  $\text{Pt}(\mathcal{S}_T)$ .

2.4.3. *Real numbers.* Let  $\mathbb{Q}^{>0}$  denote the set of positive rational numbers. An *upper real* is a model of the geometric theory  $T_u$  over  $\mathbb{Q}^{>0}$  with the following axioms:

$$\begin{aligned} q &\vdash q' & (q \leq q'), \\ q &\vdash \bigvee_{q' < q} q'. \end{aligned}$$

That is, an upper real is a subset  $U \subseteq \mathbb{Q}^{>0}$  such that

$$q \in U \iff (\exists q' < q) q' \in U.$$

The class of the upper reals will be denoted by  $\mathbb{R}^u$ , and the formal topology determined by the theory of the upper reals will be denoted by  $\mathcal{R}^u$ .

Non-negative rational numbers are embedded into  $\mathbb{R}^u$  by  $r \mapsto \{q \in \mathbb{Q}^{>0} \mid r < q\}$ , which we simply write as  $r$ . The orders on  $\mathbb{R}^u$  is defined by

$$\begin{aligned} U \leq V &\stackrel{\text{def}}{\iff} V \subseteq U, \\ U < V &\stackrel{\text{def}}{\iff} (\exists r \in \mathbb{Q}^{>0}) U + r \leq V. \end{aligned}$$

Note that for  $U \in \mathbb{R}^u$  and  $q \in \mathbb{Q}^{>0}$ , we have  $U < q \iff q \in U$ .

An upper real is *finite* if it is inhabited, i.e. if it is a model of the theory  $T_u$  with the extra axiom:

$$\top \vdash \bigvee_{q \in \mathbb{Q}^{>0}} q.$$

A *non-finite Dedekind real* is a model of the geometric theory  $T_D$  over  $G_D \stackrel{\text{def}}{=} \mathbb{Q} + \mathbb{Q}$ , elements of which will be denoted by  $(p, +\infty) \stackrel{\text{def}}{=} (0, p)$  and  $(-\infty, q) \stackrel{\text{def}}{=} (1, q)$ . The axioms of  $T_D$  are the following:

$$\begin{aligned} (-\infty, q) &\vdash (-\infty, q') & (q \leq q') \\ (-\infty, q) &\vdash \bigvee_{q' < q} (-\infty, q') \\ (p, +\infty) &\vdash (p', +\infty) & (p' \leq p) \\ (p, +\infty) &\vdash \bigvee_{p' > p} (p', +\infty) \\ (p, +\infty) \wedge (-\infty, q) &\vdash \bigvee \{(p, +\infty) \wedge (-\infty, q) \mid p < q\} \\ \top &\vdash (p, +\infty) \vee (-\infty, q) & (p < q) \end{aligned}$$

A non-finite Dedekind real  $m$  is equivalent to a pair  $(L, U)$  of possibly empty lower cut  $L$  and upper cut  $U$  with the following correspondence:

$$L \stackrel{\text{def}}{=} \{p \in \mathbb{Q} \mid (p, +\infty) \in m\}, \quad U \stackrel{\text{def}}{=} \{q \in \mathbb{Q} \mid (-\infty, q) \in m\}.$$

The orders and additions on the non-finite Dedekind reals are defined by adding the conditions dual to those of the upper reals to the lower cuts. For example, if  $(L, U)$  and  $(L', U')$  are non-finite Dedekind reals  $(L, U) \leq (L', U') \stackrel{\text{def}}{\iff} L \subseteq L' \text{ \& } U' \subseteq U$ .

A *non-negative non-finite Dedekind real* is a model of the theory  $T_D$  extended with the following axioms:

$$(-\infty, q) \vdash \bigvee \{(-\infty, q) \mid 0 < q\}.$$

The class of non-negative non-finite Dedekind reals will be denoted by  $\mathbb{R}^{\geq 0}$ , and the formal topology determined by the theory of non-negative non-finite Dedekind reals will be denoted by  $\mathcal{R}^{\geq 0}$ .

A non-finite Dedekind real is *finite* (or just a Dedekind real) if both its lower and upper cuts are inhabited, i.e. if it is a model of the theory  $T_D$  with the extra axioms:

$$\top \vdash \bigvee_{q \in \mathbb{Q}} (-\infty, q), \quad \top \vdash \bigvee_{p \in \mathbb{Q}} (p, +\infty).$$

Non-negative Dedekind reals are defined similarly and its collection will be denoted by  $\mathbb{R}^{0+}$ .

The embedding  $q \in \mathbb{Q}^{>0} \mapsto (-\infty, q) \in G_D$  of generators gives rise to a formal topology map  $\iota_D: \mathcal{R}^{\geq 0} \rightarrow \mathcal{R}^u$ . It is not hard to see that  $\iota_D$  is a monomorphism.<sup>3</sup> The morphism  $\iota_D$  restricts to a morphism between formal topologies associated with the theories of finite upper reals and non-negative Dedekind reals.

If  $T_1 = (G_1, R_1)$  and  $T_2 = (G_2, R_2)$  are geometric theories, then the product  $\mathcal{S}_{T_1} \times \mathcal{S}_{T_2}$  is presented by generators  $G_1 + G_2$  and axioms

$$(k, p_0) \wedge \cdots \wedge (k, p_{n-1}) \vdash \bigvee_{i \in I} (k, p_0^i) \wedge \cdots \wedge (k, p_{n-1}^i)$$

for each axiom  $p_0 \wedge \cdots \wedge p_{n-1} \vdash \bigvee_{i \in I} p_0^i \wedge \cdots \wedge p_{n-1}^i \in R_k$  ( $k \in \{0, 1\}$ ). The injections of generators  $p \mapsto (k, p)$  ( $p \in G_k$ ) give rise to the projections  $p_k: \mathcal{S}_{T_1} \times \mathcal{S}_{T_2} \rightarrow \mathcal{S}_{T_k}$ .

The formal addition  $+: \mathcal{R}^u \times \mathcal{R}^u \rightarrow \mathcal{R}^u$  on  $\mathcal{R}^u$  is determined by a function  $f_+: \mathbb{Q}^{>0} \rightarrow \text{Pow}(\text{Fin}(\mathbb{Q}^{>0} + \mathbb{Q}^{>0}))$  given by

$$f_+(q) \stackrel{\text{def}}{=} \{ \{ (0, q_1), (1, q_2) \} \mid q_1 + q_2 < q \},$$

which induces the addition on  $\mathbb{R}^u$  by

$$U + V \stackrel{\text{def}}{=} \{ q_1 + q_2 \mid q_1 \in U \ \& \ q_2 \in V \}.$$

Similarly, the addition  $+: \mathcal{R}^{\geq 0} \times \mathcal{R}^{\geq 0} \rightarrow \mathcal{R}^{\geq 0}$  on  $\mathcal{R}^{\geq 0}$  is determined by a function  $g_+: G_D \rightarrow \text{Pow}(\text{Fin}(G_D + G_D))$  given by

$$\begin{aligned} g_+(-\infty, q) &\stackrel{\text{def}}{=} \{ \{ (0, (-\infty, q_1), (1, (-\infty, q_2)) \} \mid q_1 + q_2 < q \}, \\ g_+(p, +\infty) &\stackrel{\text{def}}{=} \{ \{ (0, (p_1, +\infty), (1, (p_2, +\infty)) \} \mid p < p_1 + p_2 \}, \end{aligned}$$

which induces the addition on  $\mathbb{R}^{\geq 0}$  by

$$(L_1, U_1) + (L_2, U_2) \stackrel{\text{def}}{=} (L_1 + L_2, U_1 + U_2).$$

It is easy to see that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{R}^{\geq 0} \times \mathcal{R}^{\geq 0} & \xrightarrow{\iota_D \times \iota_D} & \mathcal{R}^u \times \mathcal{R}^u \\ \downarrow + & & \downarrow + \\ \mathcal{R}^{\geq 0} & \xrightarrow{\iota_D} & \mathcal{R}^u \end{array}$$

The formal order  $\leq_u$  on  $\mathcal{R}^u$  is a subtopology of  $\mathcal{R}^u \times \mathcal{R}^u$  defined by adding the following axioms to the theory of  $\mathcal{R}^u \times \mathcal{R}^u$ :

$$(1, q) \vdash (0, q).$$

---

<sup>3</sup>Hints: use the last three axioms of the theory  $T_D$ .

Similarly, the order  $\leq_D$  on  $\mathcal{R}^{\geq 0}$  is a subtopology of  $\mathcal{R}^{\geq 0} \times \mathcal{R}^{\geq 0}$  defined by adding the following axioms to the theory of  $\mathcal{R}^{\geq 0} \times \mathcal{R}^{\geq 0}$ :

$$(0, (p, +\infty)) \wedge (1, (-\infty, q)) \vdash \bigvee \{ (0, (p, +\infty)) \wedge (1, (-\infty, q)) \mid p < q \}.$$

Note that those subtopologies induce the orders on  $\mathbb{R}^u$  and  $\mathbb{R}^{\geq 0}$  that have been defined before. The morphism  $\iota_D \times \iota_D: \mathcal{R}^{\geq 0} \times \mathcal{R}^{\geq 0} \rightarrow \mathcal{R}^u \times \mathcal{R}^u$  restricts to a morphism  $\iota_D \times \iota_D: \leq_D \rightarrow \leq_u$ , which we denote by the same symbol by an abuse of notation. It is easy to see that the following diagram commutes and that it is a pullback diagram:

$$\begin{array}{ccc} \leq_D & \xrightarrow{\iota_D \times \iota_D} & \leq_u \\ \downarrow & & \downarrow \\ \mathcal{R}^{\geq 0} \times \mathcal{R}^{\geq 0} & \xrightarrow{\iota_D \times \iota_D} & \mathcal{R}^u \times \mathcal{R}^u \end{array}$$

### 3. LOCALIC COMPLETION OF GENERALISED UNIFORM SPACES

**3.1. Generalised uniform spaces.** We first recall the notion of generalised metric space by Vickers [Vic05a]. In Bishop's theory of metric space [Bis67], a distance function takes values in the non-negative Dedekind reals. Vickers allowed three generalisations to the usual development of metric spaces: first, the distances need not be finite. Second, the values are taken in the upper reals (not necessarily in the Dedekind reals). Third, the distances are not assumed to be symmetric.

**Definition 3.1** ([Vic05a, Definition 3.4]). A *generalised metric* on a set  $X$  is a function  $d: X \times X \rightarrow \mathbb{R}^u$  such that

- (1)  $d(x, x) = 0$ ,
- (2)  $d(x, z) \leq d(x, y) + d(y, z)$

for all  $x, y, z \in X$ . A generalised metric  $d$  is *finite* if  $d$  takes values in the finite upper reals and  $d$  is called *Dedekind* if  $d$  factors uniquely through the non-negative (non-finite) Dedekind reals. A generalised metric  $d$  is *symmetric* if

$$d(x, y) = d(y, x)$$

for all  $x, y \in X$ .

If  $d$  and  $\rho$  are generalised metrics on a set  $X$ , we define

$$d \leq \rho \stackrel{\text{def}}{\iff} (\forall x, x' \in X) d(x, x') \leq \rho(x, x'). \quad (3.1)$$

If  $A$  is a finitely indexed set  $\{d_0, \dots, d_{n-1}\}$  of generalised metrics on a set  $X$ , then the function  $d_A: X \times X \rightarrow \mathbb{R}^u$  defined by

$$d_A(x, x') \stackrel{\text{def}}{=} \sup \{ d_i(x, x') \mid i < n \}$$

is again a generalised metric on  $X$ .

A set equipped with a generalised metric is called a *generalised metric space* (abbreviated as gms). A *homomorphism* from a gms  $(X, d)$  to another gms  $(Y, \rho)$  is a function  $f: X \rightarrow Y$  such that

$$\rho(f(x), f(x')) \leq d(x, x')$$

for each  $x, x' \in X$ . Generalised metric spaces and homomorphisms between them form a category **GMS**.

The following seems to be the most natural generalisation of the notion of gms.

**Definition 3.2.** A *generalised uniform space* (abbreviated as gus) is a set  $X$  equipped with a set  $M$  of generalised metrics on it, where  $M$  is inhabited and closed under binary sups with respect to the order  $\leq$  given by (3.1). A *homomorphism* from a gus  $(X, M)$  to a gus  $(Y, N)$  is a function  $f: X \rightarrow Y$  such that

$$(\forall \rho \in N) (\exists d \in M) [(\forall x, x' \in X) \rho(f(x), f(x')) \leq d(x, x')].$$

A gus  $(X, M)$  is called *finite* (Dedekind, symmetric) if each  $d \in M$  is finite (respectively, Dedekind, symmetric).

**Remark 3.3.** Bishop [Bis67, Chapter 4, Problems 17] defined a uniform space as a pair  $(X, M)$  of set  $X$  and a set  $M$  of pseudometrics on  $X$ , where  $M$  is not assumed be closed under binary sups.<sup>4</sup> However, we can equip  $X$  with a new set  $M'$  of pseudometrics on  $X$  given by  $M' = \{d_A \mid A \in \text{Fin}^+(M)\}$ , which is uniformly isomorphic to  $(X, M)$  in the sense of [Bis67, Chapter 4, Problem 17]. Hence, our assumption on  $M$  is compatible with Bishop's approach.

The generalised uniform spaces and homomorphisms between them form a category **GUS**. The category of generalised metric spaces **GMS** can be embedded into **GUS** by  $(X, d) \mapsto (X, \{d\})$ . Obviously, homomorphisms between gms's become homomorphisms between the corresponding gus's. We usually identify  $(X, \{d\})$  with  $(X, d)$ .

**Example 3.4.** Any function  $f: X \rightarrow \mathbb{R}$  from a set  $X$  to the finite Dedekind reals  $\mathbb{R}$  determines a pseudometric on  $X$  by

$$d_f(x, x') \stackrel{\text{def}}{=} |f(x) - f(x')|.$$

Hence, any subset of the set  $\mathbb{F}(X, \mathbb{R})$  of real valued functions on  $X$  determines a uniform structure on  $X$  (see Remark 3.3). Important examples are

- the set of pointwise continuous functions from a metric space to  $\mathbb{R}$ ;
- the set of uniform continuous functions from a compact metric space to  $\mathbb{R}$ ;
- the set of continuous functions from a locally compact metric space to  $\mathbb{R}$ .<sup>5</sup>

Those are examples of *function spaces* [Bis67, Chapter 3, Definition 8], the notion which has gained renewed interest in recent years (see [Bri12, Ish13, Pet16]).

Similar examples are obtained when we consider a seminorm on a linear space (e.g. over  $\mathbb{R}$ ), where a seminorm on a linear space  $V$  is a non-negative real valued function  $\|-\|: V \rightarrow \mathbb{R}^{0+}$  such that

$$\|av\| = |a|\|v\|, \quad \|v + w\| \leq \|v\| + \|w\| \quad (a \in \mathbb{R}, v, w \in V)$$

Any seminorm  $\|-\|: V \rightarrow \mathbb{R}^{0+}$  determines a pseudometric on  $V$  by

$$d(v, w) \stackrel{\text{def}}{=} \|v - w\|.$$

<sup>4</sup> Bishop [Bis67, Chapter 4, Problems 17] does not even impose inhabitedness on  $M$ . We decided to include the condition for a smooth development of localic completions of generalised uniform spaces. By doing so, we can also incorporate the theory of generalised metric spaces into that of generalised uniform spaces more naturally; for example, with our definition of gus, the inclusion of the category of gms's into that of gus's preserves finite limits.

<sup>5</sup> For the notions of local compactness and continuous functions used in this examples, see Definition 4.4.

Hence, any subset of the collection of seminorms on a linear space  $V$  determines a uniform structure on  $V$ . *Locally convex spaces* are particularly important examples of the structure of this kind, where a locally convex space is a pair  $(V, \mathfrak{N})$  of a linear space  $V$  and a set of seminorms  $\mathfrak{N}$  on  $V$  such that for each  $\|-\|_1, \|-\|_2 \in \mathfrak{N}$  and  $c \in \mathbb{Q}^{>0}$ , and for each seminorm  $\|-\|$  on  $V$ ,

$$[(\forall v \in V) \|v\| \leq c(\|v\|_1 + \|v\|_2)] \implies \|-\| \in \mathfrak{N}.$$

A simple example of a locally convex space is the ring  $C(X)$  of real valued continuous functions on a locally compact metric space  $X$  with the locally convex structure generated by the seminorms

$$\{\|-\|_K \mid K \subseteq X \text{ compact subset}\},$$

where  $\|f\|_K \stackrel{\text{def}}{=} \sup \{|f(x)| \mid x \in K\}$ . See [Bis67, Chapter 9, Section 5] for details on locally convex spaces.

**Example 3.5** (Non-symmetric gus). Lifting the restriction of symmetry allows us to metrise more spaces. If  $(X, d)$  is a generalised metric space, then we can define a generalised metric  $d_L$  on  $\text{Fin}(X)$  given by

$$d_L(A, B) \stackrel{\text{def}}{=} \sup_{a \in A} \inf_{b \in B} d(a, b),$$

which is called the *lower metric*. This construction, together with the upper and the Hausdorff generalised metric, is treated in detail by Vickers [Vic09]. These constructions can be naturally extended to generalised uniform spaces. It is interesting to see how much of the results in [Vic09] can be carried over to the setting of generalised uniformly spaces.

**Example 3.6** (Domain theory [Vic05a, Section 5]). The notion of generalised metric space (and generalised uniform space) and its localic completion to be defined in Section 3 provide a common generalisation of the theory of metric space and that of domain (in the sense of domain theory [AJ94]).

Let  $(P, \leq)$  be a poset. Then, we can define a generalised metric on  $P$  by

$$d(x, y) \stackrel{\text{def}}{=} \{q \in \mathbb{Q}^{>0} \mid x \leq y \vee 1 < q\}.$$

A more elaborate example is the rationals  $\mathbb{Q}$  with the following generalised metric:

$$d(x, y) \stackrel{\text{def}}{=} \{q \in \mathbb{Q}^{>0} \mid x < y + q\}.$$

See also Remark 3.8.

**3.2. Localic completions.** Given a gus  $(X, M)$ , we define the set

$$\text{Rad}(M) \stackrel{\text{def}}{=} M \times \mathbb{Q}^{>0}$$

of generalised radii parameterised by  $M$  and a set

$$U_X \stackrel{\text{def}}{=} \text{Rad}(M) \times X$$

of generalised formal balls. We write  $\mathbf{b}_d(x, \varepsilon)$  for the element  $((d, \varepsilon), x) \in U_X$ .

Define an order  $\leq_X$  and a transitive relation  $<_X$  on  $U_X$  by

$$\begin{aligned} \mathbf{b}_d(y, \delta) \leq_X \mathbf{b}_\rho(x, \varepsilon) &\stackrel{\text{def}}{\iff} \rho \leq d \ \& \ \rho(x, y) + \delta \leq \varepsilon, \\ \mathbf{b}_d(y, \delta) <_X \mathbf{b}_\rho(x, \varepsilon) &\stackrel{\text{def}}{\iff} \rho \leq d \ \& \ \rho(x, y) + \delta < \varepsilon. \end{aligned}$$

We extend the relations  $\leq_X$  and  $<_X$  to the subsets of  $U_X$  by

$$U \leq_X V \stackrel{\text{def}}{\iff} (\forall a \in U) (\exists b \in V) a \leq_X b$$

for all  $U, V \subseteq U_X$ , and similarly for  $<_X$ .

The *localic completion* of a gus  $X = (X, M)$  is a formal topology

$$\mathcal{U}(X) = (U_X, \triangleleft_X, \leq_X),$$

where  $\triangleleft_X$  is inductively generated by the axiom-set on  $U_X$  consisting of the following axioms:

- (U1)  $a \triangleleft_X \{b \in U_X \mid b <_X a\}$ ;
- (U2)  $a \triangleleft_X \mathcal{C}_d^\varepsilon$  for each  $(d, \varepsilon) \in \text{Rad}(M)$

for each  $a \in S$ , where we define

$$\mathcal{C}_d^\varepsilon \stackrel{\text{def}}{=} \{b_d(x, \varepsilon) \in U_X \mid x \in X\}. \quad (3.2)$$

**Remark 3.7.** For a generalised metric space  $(X, d)$ , the localic completion of the gus  $(X, \{d\})$  is equivalent to the localic completion  $\mathcal{M}(X, d)$  of the gms  $(X, d)$  by Vickers [Vic05a], and we use the notation  $\mathcal{M}(X, d)$  to denote  $\mathcal{U}(X, \{d\})$ .

**Remark 3.8.** One of the important aspects of the localic completion is that the notion captures important constructions on symmetric uniform spaces and domains in a single setting. For example, Vickers showed that the localic completion of the generalised metric associated with a poset  $(P, \leq)$  given in Example 3.6 represents the Scott topology on  $P$  [Vic05a, Proposition 5.6], and the localic completion of the generalised metric on  $\mathbb{Q}$  in the same example represents the topology determined by the geometric theory of the lower cuts (the dual notion of the finite upper reals defined in Section 2.4.3) [Vic05a, Proposition 5.7].

What the localic completion means for the usual symmetric case will be studied in Section 3.4. We will not pursue the domain theoretic aspect of localic completions further in this paper.

**Lemma 3.9.** *The axioms of the form (U2) are equivalent to the following axioms:*

$$(U2') \quad a \triangleleft_X \mathcal{C}_d^\varepsilon \downarrow a \text{ for each } (d, \varepsilon) \in \text{Rad}(M),$$

*that is, together with (U1), they generate the same cover on  $U_X$ .*

*Proof.* Obvious. □

Note that the axiom-set consisting of axioms of the forms (U1) and (U2') is localised with respect to  $\leq_X$ .

For each  $b_d(x, \varepsilon) \in U_X$ , we write  $b_d(x, \varepsilon)_*$  or  $B_d(x, \varepsilon)$  to denote the *open ball* corresponding to  $b_d(x, \varepsilon)$ , i.e.

$$b_d(x, \varepsilon)_* \stackrel{\text{def}}{=} B_d(x, \varepsilon) \stackrel{\text{def}}{=} \{x' \in X \mid d(x, x') < \varepsilon\}.$$

We extend the notation  $(-)_*$  to the subsets of  $U_X$  by  $U_* \stackrel{\text{def}}{=} \bigcup_{a \in U} a_*$ .

Dually, each  $x \in X$  is associated with the set  $\diamond x$  of *open neighbourhoods* of  $x$ , namely

$$\diamond x \stackrel{\text{def}}{=} \{a \in U_X \mid x \in a_*\}.$$

**Lemma 3.10.** *Let  $(X, M)$  be a gus. Then*

- (1)  $a' \leq_X a <_X b \leq_X b' \implies a' <_X b'$ ,
- (2)  $a <_X b \implies (\exists c \in U_X) a <_X c <_X b$ ,



(3)  $a \leq_X b \implies a_* \subseteq b_*$   
for all  $a, a', b, b' \in U_X$ .

*Proof.* Straightforward.  $\square$

**Remark 3.11.** The converse of Lemma 3.10 (3) need not hold. For example, consider the unit interval  $([0, 1], d)$  of  $\mathbb{R}$ , where  $d$  denotes the usual metric on  $[0, 1]$ . We have  $B_d(1, 3) \subseteq B_d(1, 2)$ , but  $\mathbf{b}_d(1, 3) \leq_{[0, 1]} \mathbf{b}_d(1, 2)$  is false.

**Proposition 3.12.** For any gus  $X$ , its localic completion  $\mathcal{U}(X)$  is overt, and the base  $U_X$  is the positivity of  $\mathcal{U}(X)$ .

*Proof.* Straightforward.  $\square$

**3.3. Functoriality of localic completions.** Given a homomorphism  $f: (X, M) \rightarrow (Y, N)$  of gus's, define a relation  $r_f \subseteq U_X \times U_Y$  by

$$\mathbf{b}_d(x, \varepsilon) r_f \mathbf{b}_\rho(y, \delta) \stackrel{\text{def}}{\iff} d \omega_f \rho \ \& \ \mathbf{b}_\rho(f(x), \varepsilon) <_Y \mathbf{b}_\rho(y, \delta), \quad (3.3)$$

where  $d \omega_f \rho \stackrel{\text{def}}{\iff} (\forall x, x' \in X) \rho(f(x), f(x')) \leq d(x, x')$ .

**Proposition 3.13.** The localic completion extends to a functor from **GUS** to **FTop**.

*Proof.* We must show that the assignment  $f \mapsto r_f$  is functorial. First, we show that  $r_f$  is a formal topology map for any homomorphism  $f: (X, M) \rightarrow (Y, N)$ . We show that  $r_f$  satisfies (FTM2). The other properties of the morphism are easy to prove.

(FTM2) Suppose that  $\mathbf{b}_d(x, \varepsilon) \in r_f^- \mathbf{b}_{\rho_1}(y_1, \delta_1) \downarrow r_f^- \mathbf{b}_{\rho_2}(y_2, \delta_2)$ . Then,

$$\begin{aligned} d \omega_f \rho_1 \ \& \ \mathbf{b}_{\rho_1}(f(x), \varepsilon) <_Y \mathbf{b}_{\rho_1}(y_1, \delta_1), \\ d \omega_f \rho_2 \ \& \ \mathbf{b}_{\rho_2}(f(x), \varepsilon) <_Y \mathbf{b}_{\rho_2}(y_2, \delta_2). \end{aligned}$$

Let  $\rho = \sup \{\rho_1, \rho_2\}$ , and choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\mathbf{b}_{\rho_1}(f(x), \varepsilon + \theta) <_Y \mathbf{b}_{\rho_1}(y_1, \delta_1)$  and  $\mathbf{b}_{\rho_2}(f(x), \varepsilon + \theta) <_Y \mathbf{b}_{\rho_2}(y_2, \delta_2)$ . Then,  $\mathbf{b}_\rho(f(x), \varepsilon + \theta) \in \mathbf{b}_{\rho_1}(y_1, \delta_1) \downarrow \mathbf{b}_{\rho_2}(y_2, \delta_2)$ . Moreover,  $\rho(f(x), f(x')) \leq d(x, x')$  for all  $x, x' \in X$ , so that  $d \omega_f \rho$ . Thus,  $\mathbf{b}_d(x, \varepsilon) r_f \mathbf{b}_\rho(f(x), \varepsilon + \theta)$ , from which (FTM2) follows.

Next we show that the assignment is functorial. First, for any gus  $(X, M)$ , we have  $\mathbf{b}_d(x, \varepsilon) r_{\text{id}_X} \mathbf{b}_\rho(y, \delta) \iff \mathbf{b}_d(x, \varepsilon) <_X \mathbf{b}_\rho(y, \delta)$  so that  $r_{\text{id}_X} = \text{id}_{\mathcal{U}(X)}$  as formal topology maps. Second, let  $f: (X, M) \rightarrow (Y, N)$  and  $g: (Y, N) \rightarrow (Z, L)$  be homomorphisms. Let  $a = \mathbf{b}_d(x, \varepsilon) \in U_X$  and  $c = \mathbf{b}_\rho(z, \xi) \in U_Z$ , and suppose that  $\mathbf{b}_d(x, \varepsilon) r_{g \circ f} \mathbf{b}_\rho(z, \xi)$ . Then,  $d \omega_{g \circ f} \rho \ \& \ \mathbf{b}_\rho(g(f(x)), \varepsilon) < \mathbf{b}_\rho(z, \xi)$ . Since  $f$  and  $g$  are homomorphisms, there exist  $\rho' \in N$  and  $d' \in M$  such that  $d' \omega_f \rho' \omega_g \rho$ . Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\mathbf{b}_\rho(g(f(x)), \varepsilon + \theta) <_Z \mathbf{b}_\rho(z, \xi)$ . Let  $\mathbf{b}_{d^*}(x', \varepsilon') \in a \downarrow \mathcal{C}_{d'}^\theta$ . Then, we have  $\mathbf{b}_{d^*}(x', \varepsilon') r_f \mathbf{b}_{\rho'}(f(x'), \varepsilon' + \theta) r_g \mathbf{b}_\rho(z, \xi)$ . Thus  $r_{g \circ f}^- c \triangleleft_X (r_g \circ r_f)^- c$ . We also easily have  $(r_g \circ r_f)^- c \triangleleft_X r_{g \circ f}^- c$ . Hence  $r_{g \circ f} = r_g \circ r_f$ .  $\square$

Let us denote this functor by  $\overline{(-)}: \mathbf{GUS} \rightarrow \mathbf{FTop}$ .

The binary product of gus's  $(X, M)$  and  $(Y, N)$  is defined by

$$X \times Y = (X \times Y, M \times N),$$

where an element  $(d, \rho) \in M \times N$  is regarded as a generalised metric on  $X \times Y$  defined by

$$(d, \rho)((x, y), (x', y')) \stackrel{\text{def}}{=} \max \{d(x, x'), \rho(y, y')\}.$$

Binary sups in  $M \times N$  is defined coordinate-wise. The projections  $\pi_X: X \times Y \rightarrow X$  and  $\pi_Y: X \times Y \rightarrow Y$  are obviously homomorphisms of gus's. A terminal gus is  $(\{*\}, \{d_*\})$ , which is a one-point set with discrete metric  $d_*$ .

**Proposition 3.14.** *The functor  $\overline{(-)}: \mathbf{GUS} \rightarrow \mathbf{FTop}$  preserves finite products.*

*Proof.* The proof of the preservation of a terminal object is the same as that of generalised metric spaces [Vic05a, Proposition 5.3].

We sketch the proof of preservation of binary products of gus's, which is analogous to the corresponding fact about gms's [Vic05a, Theorem 5.4]. Given gus's  $(X, M)$  and  $(Y, N)$ , the functor sends the projection  $\pi_X: X \times Y \rightarrow X$  to a formal topology map  $\overline{\pi_X}: \mathcal{U}(X \times Y) \rightarrow \mathcal{U}(X)$ , which can be easily shown to be equal to a relation  $r_X \subseteq U_{X \times Y} \times U_X$  defined by

$$\mathbf{b}_{(d, \rho)}((x, y), \xi) r_X \mathbf{b}_{d'}(x', \varepsilon') \stackrel{\text{def}}{\iff} \mathbf{b}_d(x, \xi) <_X \mathbf{b}_{d'}(x', \varepsilon').$$

There is also a relation  $r_Y \subseteq U_{X \times Y} \times U_Y$ , similarly defined as  $r_X$ , which is equal to  $\overline{\pi_Y}$ . On the other hand, we define a relation  $r \subseteq (U_X \times U_Y) \times U_{X \times Y}$  by

$$(\mathbf{b}_d(x, \varepsilon), \mathbf{b}_\rho(y, \delta)) r \mathbf{b}_{(d', \rho')}((x', y'), \xi) \stackrel{\text{def}}{\iff} \mathbf{b}_d(x, \varepsilon) <_X \mathbf{b}_{d'}(x', \xi) \ \& \ \mathbf{b}_\rho(y, \delta) <_Y \mathbf{b}_{\rho'}(y', \xi),$$

which can be easily shown to be a formal topology map.

Then, it is straightforward to show that  $\langle r_X, r_Y \rangle \circ r = \text{id}_{\mathcal{U}(X) \times \mathcal{U}(Y)}$  and  $r \circ \langle r_X, r_Y \rangle = \text{id}_{\mathcal{U}(X \times Y)}$ .  $\square$

The product of a set-indexed family  $(X_i)_{i \in I}$  of gus's, each member of which is of the form  $X_i = (X_i, M_i)$ , consists of the cartesian product  $\prod_{i \in I} X_i$  and the set

$$M_\Pi \stackrel{\text{def}}{=} \text{Fin}^+ \left( \sum_{i \in I} M_i \right)$$

of generalised metrics on  $\prod_{i \in I} X_i$ , where we identify each member  $A \in M_\Pi$  with a generalised metric on  $\prod_{i \in I} X_i$  defined by

$$A(f, g) \stackrel{\text{def}}{=} \sup \{d(f(i), g(i)) \mid (i, d) \in A\}.$$

Each projection  $\pi_i: \prod_{i \in I} X_i \rightarrow X_i$  is obviously a homomorphism. We say that the product of a family  $(X_i)_{i \in I}$  is *inhabited* if the underlying set  $\prod_{i \in I} X_i$  is inhabited.

**Proposition 3.15.** *The functor  $\overline{(-)}: \mathbf{GUS} \rightarrow \mathbf{FTop}$  preserves inhabited countable products.*

*Proof.* Let  $((X_n, M_n))_{n \in \mathbb{N}}$  be a sequence of gus's with a chosen sequence  $\varphi \in \prod_{n \in \mathbb{N}} X_n$ . Let  $\prod_{n \in \mathbb{N}} X_n$  denote the product of the family, where we left the underlying family of generalised metrics implicit. Write  $\mathcal{U}(\prod_{n \in \mathbb{N}} X_n) = (U_X, \triangleleft_X, \leq_X)$  for its localic completion. The elements of  $\prod_{n \in \mathbb{N}} X_n$  will be denoted by Greek letters  $\alpha, \beta, \gamma$ , and we write  $\alpha_n$  for  $\alpha(n)$ .

Given a sequence  $(r_n: \mathcal{S} \rightarrow \mathcal{U}(X_n))_{n \in \mathbb{N}}$  of formal topology maps, defined a relation  $r \subseteq S \times U_X$  by

$$a \, r \, \mathbf{b}_A(\alpha, \varepsilon) \stackrel{\text{def}}{\iff} (\exists \mathbf{b}_B(\beta, \delta) <_X \mathbf{b}_A(\alpha, \varepsilon)) (\forall (i, d) \in B) a \triangleleft r_i^- \mathbf{b}_d(\beta_i, \delta).$$

We claim that  $r$  is a formal topology map. We only show that  $r$  satisfies (FTM2) since other conditions are easy to check. Let  $\mathbf{b}_A(\alpha, \varepsilon), \mathbf{b}_B(\beta, \delta) \in U_X$ , and let  $a \in r^- \mathbf{b}_A(\alpha, \varepsilon) \downarrow r^- \mathbf{b}_B(\beta, \delta)$ . Then, there exist  $\mathbf{b}_{A'}(\alpha', \varepsilon') <_X \mathbf{b}_A(\alpha, \varepsilon)$  and  $\mathbf{b}_{B'}(\beta', \delta') <_X \mathbf{b}_B(\beta, \delta)$  such that

- $(\forall (i, d) \in A') a \triangleleft r_i^- \mathbf{b}_d(\alpha'_i, \varepsilon')$ ,
- $(\forall (j, \rho) \in B') a \triangleleft r_j^- \mathbf{b}_\rho(\beta'_j, \delta')$ .

We can write  $A'$  as a disjoint union

$$A' = (\{i_0\} \times A_0) \cup \cdots \cup (\{i_n\} \times A_n),$$

where for each  $k \leq n$ , we have  $i_k \in \mathbb{N}$  and  $A_k \in \text{Fin}^+(M_{i_k})$ , and  $0 \leq k < k' \leq n \implies i_k \neq i_{k'}$ . Similarly, write  $B'$  as a disjoint union  $B' = (\{j_0\} \times B_0) \cup \cdots \cup (\{j_m\} \times B_m)$  that satisfies the analogous properties as those of  $A'$ . Put  $I = \{i_k \mid k \leq n\}$ ,  $J = \{j_k \mid k \leq m\}$ , and  $P = I \cup J = \{p_0, \dots, p_l\}$  with the property  $0 \leq k < k' \leq l \implies p_k \neq p_{k'}$ .

Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $A(\alpha, \alpha') + \varepsilon' + 2\theta < \varepsilon$ , and  $B(\beta, \beta') + \delta' + 2\theta < \delta$ . For each  $k \leq l$ , define a subset  $V_k \subseteq U_{X_{p_k}}$  by cases:

- (1) If  $p_k \in I \cap J$ , then put  $V_k \stackrel{\text{def}}{=} V(A_{p_k}) \downarrow V(B_{p_k}) \downarrow \mathcal{C}_{\sup(A_{p_k} \cup B_{p_k})}^\theta$ . Here, the subset  $V(A_{p_k}) \subseteq U_{p_k}$  is defined by

$$V(A_{p_k}) \stackrel{\text{def}}{=} \mathbf{b}_{d_0}(\alpha'_{p_k}, \varepsilon') \downarrow \cdots \downarrow \mathbf{b}_{d_N}(\alpha'_{p_k}, \varepsilon'),$$

where  $A_{p_k} = \{d_0, \dots, d_N\}$ . The subset  $V(B_{p_k}) \subseteq U_{p_k}$  is defined similarly.

- (2) If  $p_k \in I \setminus J$ , put  $V_k \stackrel{\text{def}}{=} V(A_{p_k}) \downarrow \mathcal{C}_{\sup(A_{p_k})}^\theta$ .
- (3) If  $p_k \in J \setminus I$ , put  $V_k \stackrel{\text{def}}{=} V(B_{p_k}) \downarrow \mathcal{C}_{\sup(B_{p_k})}^\theta$ .

Then, we have  $a \triangleleft r_{p_0}^- V_0 \downarrow \cdots \downarrow r_{p_l}^- V_l$ . Let  $b \in r_{p_0}^- V_0 \downarrow \cdots \downarrow r_{p_l}^- V_l$ . Then, for each  $k \leq l$ , there exists  $\mathbf{b}_{\sigma_k}(z_k, \xi_k) \in V_k$  such that  $b \triangleleft r_{p_k}^- \mathbf{b}_{\sigma_k}(z_k, \xi_k)$ . Define  $\gamma \in \prod_{n \in \mathbb{N}} X_n$  by

$$\gamma_n = \begin{cases} z_k & n = p_k \text{ for some } k \leq l, \\ \varphi_n & \text{otherwise,} \end{cases}$$

and put  $C = \{(p_k, \sigma_k) \mid k \leq l\}$ . Then, we have  $b \, r \, \mathbf{b}_C(\gamma, 2\theta)$ . Moreover,

$$\begin{aligned} A(\alpha, \gamma) + 2\theta &\leq A(\alpha, \alpha') + A(\alpha', \gamma) + 2\theta \\ &\leq A(\alpha, \alpha') + A'(\alpha', \gamma) + 2\theta \\ &< A(\alpha, \alpha') + \varepsilon' + 2\theta < \varepsilon. \end{aligned}$$

Thus  $\mathbf{b}_C(\gamma, 2\theta) <_X \mathbf{b}_A(\alpha, \varepsilon)$ . Similarly we have  $\mathbf{b}_C(\gamma, 2\theta) <_X \mathbf{b}_B(\beta, \delta)$ . Hence,

$$a \triangleleft r^- (\mathbf{b}_A(\alpha, \varepsilon) \downarrow \mathbf{b}_B(\beta, \delta)).$$

Next, we note that the following holds:

$$a \, r \, \mathbf{b}_A(\alpha, \varepsilon) \implies (\forall (i, d) \in A) a \triangleleft r_i^- \mathbf{b}_d(\alpha_i, \varepsilon).$$

The proof is similar to the above proof of the condition (FTM2) for  $r$ . Moreover, the functor  $\overline{(-)}: \mathbf{GUS} \rightarrow \mathbf{FTop}$  sends each projection  $\pi_n: \prod_{n \in \mathbb{N}} X_n \rightarrow X_n$  to a formal topology map  $\overline{\pi_n}: \mathcal{U}(\prod_{n \in \mathbb{N}} X_n) \rightarrow \mathcal{U}(X_n)$ . Clearly, we have the following:

$$\mathbf{b}_A(\alpha, \varepsilon) \overline{\pi_n} \mathbf{b}_d(x, \delta) \iff \mathbf{b}_A(\alpha, \varepsilon) <_X \mathbf{b}_{\{(n,d)\}}(\varphi_{(n,x)}, \delta),$$

where  $\varphi_{(n,x)}$  is obtained from  $\varphi$  by replacing  $n$ th element with  $x$ .

With these at our disposal, it is straightforward to show that  $r$  makes the diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{r} & \mathcal{U}(\prod_{n \in \mathbb{N}} X_n) \\ & \searrow r_n & \downarrow \overline{\pi_n} \\ & & \mathcal{U}(X_n) \end{array}$$

commute, and it is a unique such morphism.  $\square$

Note that Proposition 3.15 generalises the fact that the localic completion of Baire space  $\mathbb{N}^{\mathbb{N}}$  is the point-free Baire space [Pal14, Proposition 3.1].

Let  $(X, M)$  be a gus, and let  $M^{\text{op}}$  denote  $M$  ordered by the opposite of (3.1). Then, we have a cofiltered diagram  $D_M: M^{\text{op}} \rightarrow \mathbf{GUS}$  given by  $D_M(d) = (X, d)$  for  $d \in M$ , and for each  $d, d' \in M$  such that  $d \leq d'$  we have  $D_M(d, d') \stackrel{\text{def}}{=} \text{id}_X: (X, d') \rightarrow (X, d)$ . For each  $d \in M$ , we have a homomorphism  $\sigma_d: (X, M) \rightarrow (X, d)$  with the identity function  $\text{id}_X$  as the underlying map. It is easy to see that the family  $(\sigma_d: (X, M) \rightarrow (X, d))_{d \in M}$  is a limit of the diagram  $D_M$ .

**Proposition 3.16.** *For any gus  $(X, M)$ , the functor  $\overline{(-)}: \mathbf{GUS} \rightarrow \mathbf{FTop}$  preserves the limit of the cofiltered diagram  $D_M: M^{\text{op}} \rightarrow \mathbf{GUS}$ .*

*Proof.* We write  $\mathcal{U}(X)$  for the localic completion of  $(X, M)$  and  $\mathcal{M}(X, d)$  for the localic completion of  $(X, d)$  for each  $d \in M$ . It suffices to show that the family

$$(\overline{\sigma_d}: \mathcal{U}(X) \rightarrow \mathcal{M}(X, d))_{d \in M}$$

is a limit of the diagram  $\overline{(-)} \circ D_M: M^{\text{op}} \rightarrow \mathbf{FTop}$ .

First, for each  $d, d' \in M$  such that  $d \leq d'$ , we have

$$\mathbf{b}_{d'}(x', \varepsilon') \overline{D_M(d, d')} \mathbf{b}_d(x, \varepsilon) \iff \mathbf{b}_{d'}(x', \varepsilon') <_X \mathbf{b}_d(x, \varepsilon).$$

Moreover, for each  $d \in M$ , we have

$$\mathbf{b}_{d'}(x', \varepsilon') \overline{\sigma_d} \mathbf{b}_d(x, \varepsilon) \iff \mathbf{b}_{d'}(x', \varepsilon') <_X \mathbf{b}_d(x, \varepsilon).$$

Given any cone  $(r_d: \mathcal{S} \rightarrow \mathcal{M}(X, d))_{d \in M}$  over  $\overline{(-)} \circ D_M: M^{\text{op}} \rightarrow \mathbf{FTop}$ , define a relation  $r \subseteq \mathcal{S} \times U_X$  by

$$a \, r \, \mathbf{b}_d(x, \varepsilon) \stackrel{\text{def}}{\iff} a \, r_d \, \mathbf{b}_d(x, \varepsilon)$$

for all  $a \in \mathcal{S}$  and  $\mathbf{b}_d(x, \varepsilon) \in U_X$ . It is straightforward to show that  $r$  is a formal topology map  $r: \mathcal{S} \rightarrow \mathcal{U}(X)$ . For example, the property (FTM1) follows from the fact that  $M$  is inhabited. For the property (FTM2), let  $\mathbf{b}_{d_1}(x_1, \varepsilon_1), \mathbf{b}_{d_2}(x_2, \varepsilon_2) \in U_X$ . Putting  $d = \sup\{d_1, d_2\}$ , we have

$$\begin{aligned} r^- \mathbf{b}_{d_1}(x_1, \varepsilon_1) \downarrow r^- \mathbf{b}_{d_2}(x_2, \varepsilon_2) &\triangleleft r_d^- \overline{D_M(d_1, d)}^- \mathbf{b}_{d_1}(x_1, \varepsilon_1) \downarrow r_d^- \overline{D_M(d_2, d)}^- \mathbf{b}_{d_2}(x_2, \varepsilon_2) \\ &\triangleleft r_d^- \left( \overline{D_M(d_1, d)}^- \mathbf{b}_{d_1}(x_1, \varepsilon_1) \downarrow \overline{D_M(d_2, d)}^- \mathbf{b}_{d_2}(x_2, \varepsilon_2) \right) \\ &\triangleleft r^- (\mathbf{b}_{d_1}(x_1, \varepsilon_1) \downarrow \mathbf{b}_{d_2}(x_2, \varepsilon_2)). \end{aligned}$$

The other properties of  $r$  are easy to check. Then, it is straightforward to show that  $\overline{\sigma_d} \circ r = r_d$  for each  $d \in M$ , and that  $r$  is the unique formal topology map with this property.  $\square$

**3.4. Symmetric generalised uniform spaces.** We fix a symmetric gus  $(X, M)$  throughout this subsection. The aims of this subsection are twofold. The first is to obtain a point-free analogue of the point-set completion of  $(X, M)$  as a closed subspace of the product of the completion of  $(X, d)$  for each  $d \in M$  (Section 3.4.1). The second is to analyse the point-free uniform structure on  $\mathcal{U}(X)$  induced by each symmetric generalised metric  $d \in M$  (Section 3.4.2), and relate it to the complete uniform structure on  $\text{Pt}(\mathcal{U}(X))$  (Section 3.4.3).

**3.4.1. Closed embedding into  $\prod_{d \in M} \mathcal{M}(X, d)$ .** Since  $\mathcal{U}(X)$  is a limit of the diagram  $\overline{(-)} \circ D_M: M^{\text{op}} \rightarrow \mathbf{FTop}$ , the unique morphism  $\iota_{\mathcal{U}(X)}: \mathcal{U}(X) \rightarrow \prod_{d \in M} \mathcal{M}(X, d)$  determined by the family  $(\overline{\sigma_d}: \mathcal{U}(X) \rightarrow \mathcal{M}(X, d))_{d \in M}$  is an embedding (because it must be an equaliser). The image of  $\iota_{\mathcal{U}(X)}$  is overt with positivity

$$\iota_{\mathcal{U}(X)}[U_X] = \left\{ A \in \text{Fin}\left(\sum_{d \in M} U_{(X, d)}\right) \mid (\exists a \in U_X) (\forall (d, b) \in A) a \triangleleft_X b \right\}.$$

**Lemma 3.17.**  $\iota_{\mathcal{U}(X)}[U_X] = \{A \in \text{Fin}(\sum_{d \in M} U_{(X, d)}) \mid (\exists x \in X) (\forall (d, a) \in A) x \in a_*\}$ .

*Proof.* The inclusion  $\subseteq$  is clear. Conversely, let  $A \in \text{Fin}(\sum_{d \in M} U_{(X, d)})$ , and let  $x \in X$  such that  $x \in a_*$  for each  $(d, a) \in A$ . Without loss of generality, we can assume that  $A$  is inhabited. Write  $A = \{(d_0, \mathbf{b}_{d_0}(x_0, \varepsilon_0)), \dots, (d_n, \mathbf{b}_{d_n}(x_n, \varepsilon_n))\}$ , and choose  $\theta \in \mathbb{Q}^{>0}$  such that  $d_i(x_i, x) + \theta < \varepsilon_i$  for each  $i \leq n$ . Let  $d = \sup\{d_i \mid i \leq n\}$ . Then,  $\mathbf{b}_d(x, \theta) <_X \mathbf{b}_{d_i}(x_i, \varepsilon_i)$  for each  $i \leq n$ . Hence  $A \in \iota_{\mathcal{U}(X)}[U_X]$ .  $\square$

Define  $W \subseteq \text{Fin}(\sum_{d \in M} U_{(X, d)})$  by  $W \stackrel{\text{def}}{=} \iota_{\mathcal{U}(X)}[U_X]$ . Let  $\mathcal{S}_W = (S_W, \triangleleft_W \leq)$  be the overt weakly closed subtopology of  $\prod_{d \in M} \mathcal{M}(X, d)$  determined by  $W$ , where  $\leq$  is the pre-order on the base of  $\prod_{d \in M} \mathcal{M}(X, d)$ .

**Lemma 3.18.** For each  $d, d' \in M$  such that  $d \leq d'$ , the diagram commutes:

$$\begin{array}{ccc} & \mathcal{S}_W & \\ p_{d'} \swarrow & & \searrow p_d \\ \mathcal{M}(X, d') & \xrightarrow{\overline{D_M(d, d')}} & \mathcal{M}(X, d) \end{array}$$

where  $p_d$  (respectively  $p_{d'}$ ) is the restriction of the projections  $p_d: \prod_{d \in M} \mathcal{M}(X, d) \rightarrow \mathcal{M}(X, d)$  to  $\mathcal{S}_W$ .

*Proof.* Let  $\mathbf{b}_d(x, \varepsilon) \in U_{(X, d)}$  and  $A \in \mathcal{S}_W$ .

First, suppose that  $A p_d \mathbf{b}_d(x, \varepsilon)$ , i.e.  $A = \{(d, \mathbf{b}_d(x, \varepsilon))\}$ . We must show that

$$A \triangleleft_W \{ \{(d', \mathbf{b}_{d'}(y, \delta))\} \mid \mathbf{b}_{d'}(y, \delta) <_X \mathbf{b}_d(x, \varepsilon) \}.$$

By (U1), we have

$$A \triangleleft_W \{ \{(d, \mathbf{b}_d(x, \varepsilon'))\} \mid \varepsilon' < \varepsilon \}.$$

Let  $\varepsilon' \in \mathbb{Q}^{>0}$  such that  $\varepsilon' < \varepsilon$ , and choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\varepsilon' + 2\theta < \varepsilon$ . By (U2), we have

$$\{(d, \mathbf{b}_d(x, \varepsilon'))\} \triangleleft_W \{ \{(d, \mathbf{b}_d(x, \varepsilon')), (d', \mathbf{b}_{d'}(y, \theta))\} \mid y \in X \} \cap W.$$

Let  $y \in X$ , and suppose that  $\{(d, \mathbf{b}_d(x, \varepsilon')), (d', \mathbf{b}_{d'}(y, \theta))\} \in W$ . Then, there exists  $z \in X$  such that  $d'(y, z) < \theta$  and  $d(x, z) < \varepsilon'$ . Thus,  $d(x, y) + \theta \leq d(x, z) + d(z, y) + \theta < d(x, z) + d'(y, z) + \theta < \varepsilon' + 2\theta < \varepsilon$ . Hence  $\{(d, \mathbf{b}_d(x, \varepsilon')), (d', \mathbf{b}_{d'}(y, \theta))\} \leq \{(d', \mathbf{b}_{d'}(y, \theta))\}$  and  $\mathbf{b}_{d'}(y, \theta) <_X \mathbf{b}_d(x, \varepsilon)$ .

The proof of the converse, i.e.  $A \overline{(D_M(d, d') \circ p_{d'})} \mathbf{b}_d(x, \varepsilon) \implies A \triangleleft_W p_d^- \mathbf{b}_d(x, \varepsilon)$  is similar.  $\square$

By Proposition 3.16, there exists a unique morphism  $s: \mathcal{S}_W \rightarrow \mathcal{U}(X)$  such that  $p_d = \overline{\sigma_d} \circ s$  for each  $d \in M$ . By the proof of Proposition 3.16, we have

$$A s \mathbf{b}_d(x, \varepsilon) \iff A \triangleleft_W \{(d, \mathbf{b}_d(x, \varepsilon))\}.$$

It is straightforward to show that  $\iota_{\mathcal{U}(X)} \circ s = \text{id}_{\mathcal{S}_W}$ . Since  $\iota_{\mathcal{U}(X)}$  is a monomorphism, it is an isomorphism.

**Theorem 3.19.** *The localic completion  $\mathcal{U}(X)$  embeds into  $\prod_{d \in M} \mathcal{M}(X, d)$  as an overt weakly closed subtopology.*

**3.4.2. Point-free uniform structures.** Vickers [Vic05a, Proposition 6.6] showed that each symmetric generalised metric  $d \in M$  determines a morphism  $\overline{d}: \mathcal{M}(X, d) \times \mathcal{M}(X, d) \rightarrow \mathcal{R}^u$  of formal topologies determined by a function  $f_d: \mathbb{Q}^{>0} \rightarrow \mathbf{Pow}(U_{(X, d)} \times U_{(X, d)})$  given by

$$f_d(q) \stackrel{\text{def}}{=} \{(\mathbf{b}_d(x_1, \varepsilon_1), \mathbf{b}_d(x_2, \varepsilon_2)) \mid d(x_1, x_2) + \varepsilon_1 + \varepsilon_2 < q\}.$$

He showed that  $\overline{d}$  satisfies the properties of point-free symmetric generalised metrics.<sup>6</sup> This means that

- (1)  $\overline{d} \circ \Delta_{\mathcal{M}(X, d)} = \mathbf{0}$ ,
- (2)  $\overline{d} \circ \tau_{\mathcal{M}(X, d)} = \overline{d}$ ,
- (3)  $\langle \overline{d} \circ \langle p_1 \circ q_1, p_2 \circ q_2 \rangle, + \circ \langle \overline{d} \circ q_1, \overline{d} \circ q_2 \rangle \rangle$  factors uniquely through the embedding  $\leq_u \rightarrow \mathcal{R}^u \times \mathcal{R}^u$ .

Here

- $\Delta_{\mathcal{M}(X, d)}$  is the diagonal morphism  $\langle \text{id}_{\mathcal{M}(X, d)}, \text{id}_{\mathcal{M}(X, d)} \rangle: \mathcal{M}(X, d) \rightarrow \mathcal{M}(X, d) \times \mathcal{M}(X, d)$ ;
- $\mathbf{0}: \mathcal{M}(X, d) \rightarrow \mathcal{R}^u$  is determined by a function  $f_{\mathbf{0}}: \mathbb{Q}^{>0} \rightarrow \mathbf{Pow}(U_{(X, d)})$  given by  $f_{\mathbf{0}}(q) = U_{(X, d)}$  for all  $q \in \mathbb{Q}^{>0}$ ;
- $\tau_{\mathcal{M}(X, d)}$  is the twisting morphism  $\langle p_2, p_1 \rangle: \mathcal{M}(X, d) \times \mathcal{M}(X, d) \rightarrow \mathcal{M}(X, d) \times \mathcal{M}(X, d)$ , where  $p_1$  and  $p_2$  are the first and second projections of  $\mathcal{M}(X, d) \times \mathcal{M}(X, d)$ ;
- $q_1, q_2: \mathcal{T} \rightarrow \mathcal{M}(X, d) \times \mathcal{M}(X, d)$  are the pullback:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{q_2} & \mathcal{M}(X, d) \times \mathcal{M}(X, d) \\ q_1 \downarrow & & \downarrow p_1 \\ \mathcal{M}(X, d) \times \mathcal{M}(X, d) & \xrightarrow{p_2} & \mathcal{M}(X, d) \end{array}$$

<sup>6</sup>Vickers's proof is in the setting of impredicative topos theory, but it is straightforward to adapt his proof to the setting of formal topology.

Moreover, Vickers showed that if  $d$  is Dedekind, then  $\bar{d}$  uniquely factors through the Dedekind reals  $\mathcal{R}^{\geq 0}$  via  $\iota_D: \mathcal{R}^{\geq 0} \rightarrow \mathcal{R}^u$  (his result can be easily adapted to the finite Dedekind reals). If  $\bar{d}_D: \mathcal{M}(X, d) \times \mathcal{M}(X, d) \rightarrow \mathcal{R}^{\geq 0}$  is a factorisation of  $\bar{d}$ , then by using the facts in Section 2.4, it is easy to see that  $\bar{d}_D$  satisfies the properties of point-free (non-finite) Dedekind symmetric generalised metrics on  $\mathcal{M}(X, d)$ .

Furthermore, for any formal topology map  $r: \mathcal{S} \rightarrow \mathcal{M}(X, d)$ , a straightforward diagram chasing shows that  $\bar{d} \circ r \times r$  (or  $\bar{d}_D \circ r \times r$ ) satisfies the properties of point-free (non-finite Dedekind) symmetric generalised metric on  $\mathcal{S}$ . Thus, each symmetric generalised metric  $d \in M$  gives rise to a point-free (non-finite Dedekind) symmetric generalised metric on  $\mathcal{U}(X)$  by composing  $\bar{d}: \mathcal{M}(X, d) \times \mathcal{M}(X, d) \rightarrow \mathcal{R}^u$  (or  $\bar{d}_D$ ) with the products of projection  $p_d: \prod_{d \in M} \mathcal{M}(X, d) \rightarrow \mathcal{M}(X, d)$  and the embedding  $\iota_{\mathcal{U}(X)}: \mathcal{U}(X) \rightarrow \prod_{d \in M} \mathcal{M}(X, d)$ . Let us denote this composite by

$$\tilde{d} \stackrel{\text{def}}{=} \bar{d} \circ p_d \times p_d \circ \iota_{\mathcal{U}(X)} \times \iota_{\mathcal{U}(X)}.$$

It is easy to see that  $\tilde{d}$  is determined by a function  $g_d: \mathbb{Q}^{>0} \rightarrow \mathbf{Pow}(U_X \times U_X)$  given by

$$g_d(q) \stackrel{\text{def}}{=} \{(\mathbf{b}_d(x_1, \varepsilon_1), \mathbf{b}_d(x_2, \varepsilon_2)) \in U_X \times U_X \mid d(x_1, x_2) + \varepsilon_1 + \varepsilon_2 < q\}.$$

Moreover, if we regard  $(X, M)$  as a discrete formal topology (with base  $X$  with the discrete order  $=$  and with the trivial cover  $x \triangleleft U \iff x \in U$ ), then each  $d \in M$  determines a point-free symmetric generalised metric  $d: X \times X \rightarrow \mathcal{R}^u$  on  $X$ , which is given by a function  $h_d: \mathbb{Q}^{>0} \rightarrow \mathbf{Pow}(X \times X)$  defined by

$$h_d(q) \stackrel{\text{def}}{=} \{(x, x') \in X \times X \mid d(x, x') < q\}.$$

Also, there is a formal topology map  $\iota_X: X \rightarrow \mathcal{U}(X)$  defined by

$$x \iota_X \mathbf{b}_d(x', \varepsilon) \stackrel{\text{def}}{\iff} d(x', x) < \varepsilon,$$

which is easily seen to be an isometry in the sense that  $d = \tilde{d} \circ \iota_X \times \iota_X$  for each  $d \in M$ .

**3.4.3. Formal points of localic completions.** In this subsection, we work impredicatively, assuming that  $\mathbf{Pow}(X)$  is a set for any set  $X$ . When specialised to finite Dedekind symmetric gus's, however, the results in this subsection give a predicative completion of a given gus (see Remark 3.31). In the next subsection (Section 3.4.4), we fully address the predicativity issue raised in this subsection. As in Section 3.4.2, we work on a fixed symmetric generalised uniform space  $(X, M)$ .

For each  $d \in M$ , by applying the operation  $\mathbf{Pt}(-)$  to the point-free symmetric generalised metric  $\tilde{d}: \mathcal{U}(X) \times \mathcal{U}(X) \rightarrow \mathcal{R}^u$ , we obtain a symmetric generalised metric

$$\mathbf{Pt}(\tilde{d}): \mathbf{Pt}(\mathcal{U}(X)) \times \mathbf{Pt}(\mathcal{U}(X)) \rightarrow \mathcal{R}^u$$

on  $\mathbf{Pt}(\mathcal{U}(X))$ . This is because  $\mathbf{Pt}(-)$  is a right adjoint and so it preserves all the properties of symmetric generalised metrics. By an abuse of notation, we write  $\tilde{d}$  for  $\mathbf{Pt}(\tilde{d})$ . Explicitly,  $\tilde{d}: \mathbf{Pt}(\mathcal{U}(X)) \times \mathbf{Pt}(\mathcal{U}(X)) \rightarrow \mathcal{R}^u$  is given by

$$\tilde{d}(\alpha, \beta) = \{q \in \mathbb{Q}^{>0} \mid (\exists \mathbf{b}_d(x, \varepsilon) \in \alpha) (\exists \mathbf{b}_d(y, \delta) \in \beta) d(x, y) + \varepsilon + \delta < q\}.$$

Moreover,  $\tilde{d}$  factors through (finite) Dedekind reals if  $d$  is (finite) Dedekind. Note that  $d \leq \rho \iff \tilde{d} \leq \tilde{\rho}$  for all  $d, \rho \in M$ , and  $\sup \{d, \rho\} = \sup \{\tilde{d}, \tilde{\rho}\}$ .

On the other hand, under the operation  $\text{Pt}(-)$ , the gus  $(X, M)$  as a discrete formal topology (with the formal uniform structure  $M$ ) is mapped essentially to  $(X, M)$  itself. The formal topology map  $i_X: X \rightarrow \mathcal{U}(X)$  is mapped by  $\text{Pt}(-)$  to a function  $i_X: X \rightarrow \text{Pt}(\mathcal{U}(X))$  given by

$$i_X(x) \stackrel{\text{def}}{=} \diamond x. \quad (3.4)$$

Moreover, it is an isometry, i.e.  $d = \tilde{d} \circ i_X \times i_X$  for each  $d \in M$ .

**Lemma 3.20.** *For each  $\alpha \in \text{Pt}(\mathcal{U}(X))$ , we have*

$$\mathbf{b}_d(x, \varepsilon) \in \alpha \iff \tilde{d}(\alpha, \diamond x) < \varepsilon.$$

*Proof.* Suppose that  $\mathbf{b}_d(x, \varepsilon) \in \alpha$ . By (U1), there exists  $\varepsilon' < \varepsilon$  such that  $\mathbf{b}_d(x, \varepsilon') \in \alpha$ . Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\varepsilon' + \theta < \varepsilon$ . Since  $\mathbf{b}_d(x, \theta) \in \diamond x$ , we have  $\tilde{d}(\alpha, \diamond x) < \varepsilon$ . Conversely, suppose that  $\tilde{d}(\alpha, \diamond x) < \varepsilon$ . Then, there exist  $\mathbf{b}_d(y, \delta) \in \diamond x$  and  $\mathbf{b}_d(z, \xi) \in \alpha$  such that  $d(y, z) + \delta + \xi < \varepsilon$ . Then,  $d(x, z) + \xi \leq d(x, y) + d(y, z) + \xi < \delta + d(y, z) + \xi < \varepsilon$ , and hence  $\mathbf{b}_d(z, \xi) <_X \mathbf{b}_d(x, \varepsilon)$ . Therefore,  $\mathbf{b}_d(x, \varepsilon) \in \alpha$ .  $\square$

**Proposition 3.21.** *The function  $i_X: X \rightarrow \text{Pt}(\mathcal{U}(X))$  is dense, i.e.*

$$(\forall \alpha \in \text{Pt}(\mathcal{U}(X))) (\forall (d, \varepsilon) \in \text{Rad}(M)) (\exists x \in X) \tilde{d}(\alpha, \diamond x) < \varepsilon.$$

*Proof.* By the axiom (U2) and Lemma 3.20.  $\square$

**Definition 3.22.** A symmetric gus  $(X, M)$  is *separated* if

$$[(\forall d \in M) d(x, y) = 0] \iff x = y.$$

**Proposition 3.23.** *A symmetric gus  $(X, M)$  is separated if and only if the function  $i_X: X \rightarrow \text{Pt}(\mathcal{U}(X))$  is injective.*

*Proof.* Suppose that  $(X, M)$  is separated, and assume  $i_X(x) = i_X(y)$ . Let  $d \in M$ . Then for each  $q \in \mathbb{Q}^{>0}$ , since  $\mathbf{b}_d(x, q) \in i_X(x)$ , we have  $d(x, y) < q$ . Thus,  $d(x, y) = 0$ . Hence  $x = y$ .

Conversely, suppose that  $i_X$  is injective. Let  $x, y \in X$ , and suppose that  $d(x, y) = 0$  for all  $d \in M$ . Let  $\mathbf{b}_d(z, \varepsilon) \in i_X(x)$ . Then,  $d(z, x) < \varepsilon$ , and so  $d(z, y) < \varepsilon$ . Thus,  $\mathbf{b}_d(z, \varepsilon) \in i_X(y)$ . Similarly (using symmetry), we have  $i_X(y) \subseteq i_X(x)$ . Since  $i_X$  is injective, we have  $x = y$ .  $\square$

In the rest of this subsection, we show that  $\text{Pt}(\mathcal{U}(X))$  equipped with  $\widetilde{M} \stackrel{\text{def}}{=} \{\tilde{d} \mid d \in M\}$  is a completion of  $(X, M)$ .

**Definition 3.24.** A *Cauchy filter* on a symmetric gus  $(X, M)$  is a set  $\mathcal{F}$  of subsets of  $X$  such that

- (CF1)  $U \in \mathcal{F} \implies U \not\emptyset X$ ,
- (CF2)  $U \in \mathcal{F} \ \& \ U \subseteq V \implies V \in \mathcal{F}$ ,
- (CF3)  $U, V \in \mathcal{F} \implies U \cap V \in \mathcal{F}$ ,
- (CF4)  $(\forall (d, \varepsilon) \in \text{Rad}(M)) (\exists x \in X) \mathbf{B}_d(x, \varepsilon) \in \mathcal{F}$ .

A Cauchy filter  $\mathcal{F}$  on  $X$  *converges* to a point  $x \in X$  if  $(\forall (d, \varepsilon) \in \text{Rad}(M)) \mathbf{B}_d(x, \varepsilon) \in \mathcal{F}$ . A symmetric gus is *complete* if every Cauchy filter on  $X$  converges to some point.

**Lemma 3.25.** *Let  $(X, M)$  be a symmetric gus.*



(1) If  $\mathcal{F}$  is a Cauchy filter on  $X$ , then

$$\alpha_{\mathcal{F}} \stackrel{\text{def}}{=} \{a \in U_X \mid (\exists b <_X a) b_* \in \mathcal{F}\}$$

is a formal point of  $\mathcal{U}(X)$ .

(2) If  $\alpha$  is a formal point of  $\mathcal{U}(X)$ , then

$$\mathcal{F}_{\alpha} \stackrel{\text{def}}{=} \{U \in \text{Pow}(X) \mid (\exists a \in \alpha) a_* \subseteq U\}$$

is a Cauchy filter on  $X$ . Moreover,  $\alpha_{\mathcal{F}_{\alpha}} = \alpha$ .

*Proof.* (1) For example, to see that  $\alpha_{\mathcal{F}}$  satisfies (P2), let  $a, b \in \alpha_{\mathcal{F}}$ . Then, there exist  $a' <_X a$  and  $b' <_X b$  such that  $a'_*, b'_* \in \mathcal{F}$ . Write  $a = \mathbf{b}_{d_1}(x, \varepsilon)$ ,  $b = \mathbf{b}_{d_2}(y, \delta)$ ,  $a' = \mathbf{b}_{d'_1}(x', \varepsilon')$ , and  $b' = \mathbf{b}_{d'_2}(y', \delta')$ . Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\mathbf{b}_{d'_1}(x', \varepsilon' + 3\theta) <_X a$ , and  $\mathbf{b}_{d'_2}(y', \delta' + 3\theta) <_X b$ , and put  $\rho = \sup\{d'_1, d'_2\}$ . Since  $\mathcal{F}$  is Cauchy, there exists  $z \in X$  such that  $\mathbf{B}_{\rho}(z, \theta) \in \mathcal{F}$ . Thus,  $\mathbf{B}_{\rho}(z, \theta) \not\subseteq \mathbf{B}_{d'_1}(x', \varepsilon')$  and  $\mathbf{B}_{\rho}(z, \theta) \not\subseteq \mathbf{B}_{d'_2}(y', \delta')$ . Then,

$$d_1(x, z) + 2\theta \leq d_1(x, x') + d_1(x', z) + 2\theta < d_1(x, x') + \varepsilon' + \theta + 2\theta < \varepsilon.$$

Thus,  $\mathbf{b}_{\rho}(z, 2\theta) <_X a$ , and similarly  $\mathbf{b}_{\rho}(z, 2\theta) <_X b$ . Moreover,  $\mathbf{b}_{\rho}(z, 2\theta) \in \alpha_{\mathcal{F}}$ .

(2) The claim that  $\mathcal{F}_{\alpha}$  is a Cauchy filter is obvious. To see that  $\alpha_{\mathcal{F}_{\alpha}} = \alpha$ , let  $a = \mathbf{b}_d(x, \varepsilon) \in \alpha_{\mathcal{F}_{\alpha}}$ . Then, there exists  $\mathbf{b}_{d'}(x', \varepsilon') <_X a$  such that  $\mathbf{B}_{d'}(x', \varepsilon') \in \mathcal{F}_{\alpha}$ . Thus, there exists  $\mathbf{b}_{\rho}(y, \delta) \in \alpha$  such that  $\mathbf{B}_{\rho}(y, \delta) \subseteq \mathbf{B}_{d'}(x', \varepsilon')$ . Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $d(x, x') + \varepsilon' + \theta < \varepsilon$ . Then, there exists  $c = \mathbf{b}_{\rho'}(y', \delta') \in \alpha$  such that  $c \in \mathbf{b}_{\rho}(y, \delta) \downarrow \mathcal{C}_{d'}^{\theta}$ . Then

$$d(x, y') + \delta' \leq d(x, x') + d(x', y') + \delta' < d(x, x') + \varepsilon' + \theta < \varepsilon.$$

Thus,  $c <_X a$ , and hence  $a \in \alpha$ .

Conversely, if  $a \in \alpha$ , then there exists  $b <_X a$  such that  $b \in \alpha$  by (U1). Then,  $b_* \in \mathcal{F}_{\alpha}$  and so  $a \in \alpha_{\mathcal{F}_{\alpha}}$ .  $\square$

**Lemma 3.26.** A Cauchy filter  $\mathcal{F}$  on  $(X, M)$  converges to  $x \in X$  if and only if  $\diamond x = \alpha_{\mathcal{F}}$ .

*Proof.* ‘If’ part is obvious from the definition of  $\alpha_{\mathcal{F}}$ .

For ‘only if’ part, suppose that  $\mathcal{F}$  converges to  $x \in X$ . Let  $a = \mathbf{b}_d(y, \delta) \in \diamond x$ . Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $d(y, x) + \theta < \delta$ . Then,  $\mathbf{b}_d(x, \theta) <_X a$ . Since  $\mathcal{F}$  converges to  $x$ , we have  $\mathbf{B}_d(x, \theta) \in \mathcal{F}$ , and so  $a \in \alpha_{\mathcal{F}}$ . Conversely, let  $a = \mathbf{b}_d(y, \delta) \in \alpha_{\mathcal{F}}$ . Then, there exists  $\mathbf{b}_{d'}(y', \delta') <_X a$  such that  $\mathbf{B}_{d'}(y', \delta') \in \mathcal{F}$ . Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $d(y, y') + \delta' + \theta < \delta$ . Since  $\mathcal{F}$  converges to  $x$ , we have  $\mathbf{B}_d(x, \theta) \in \mathcal{F}$ . Since  $\mathcal{F}$  is a filter, we have  $d(y, x) \leq d(y, y') + d(y', x) < d(y, y') + \delta' + \theta < \delta$ . Hence  $a \in \diamond x$ .  $\square$

**Proposition 3.27.** A symmetric gus  $(X, M)$  is complete if and only if the function  $i_X : X \rightarrow \text{Pt}(\mathcal{U}(X))$  is a surjection, i.e. for each  $\alpha \in \text{Pt}(\mathcal{U}(X))$  there exists  $x \in X$  such that  $\alpha = \diamond x$ .

*Proof.* By Lemma 3.25 and Lemma 3.26.  $\square$

**Lemma 3.28** (cf. [Pal07, Theorem 2.7]). Let  $(X, M)$  and  $(Y, N)$  be symmetric gus’s where  $M = \{d_i \mid i \in I\}$  and  $N = \{\rho_i \mid i \in I\}$  are indexed by the same set  $I$ , and which satisfies  $d_i \leq d_j \iff \rho_i \leq \rho_j$  for each  $i, j \in I$ . Let  $f : X \rightarrow Y$  be an isometry in the sense that  $d_i = \rho_i \circ f \times f$  for each  $i \in I$  with a dense image. Then, the functor  $\overline{(-)} : \mathbf{GUS} \rightarrow \mathbf{FTop}$  sends  $f$  to an isomorphism  $\overline{f} : \mathcal{U}(X) \rightarrow \mathcal{U}(Y)$ .

*Proof.* The reader is referred to a quite similar proof of [Pal07, Theorem 2.7]. Note that

$$\mathbf{b}_{d_i}(x, \varepsilon) \overline{f} \mathbf{b}_{\rho_j}(y, \delta) \iff \mathbf{b}_{\rho_i}(f(x), \varepsilon) <_Y \mathbf{b}_{\rho_j}(y, \delta). \quad \square$$

By Proposition 3.21 and Lemma 3.28, the isometry  $i_X: X \rightarrow \text{Pt}\mathcal{U}(X)$  gives rise to an isomorphism  $\overline{i_X}: \mathcal{U}(X) \rightarrow \mathcal{U}(\text{Pt}(\mathcal{U}(X)))$  defined by

$$\mathbf{b}_d(x, \varepsilon) \overline{i_X} \mathbf{b}_{\tilde{d}}(\alpha, \delta) \iff \mathbf{b}_{\tilde{d}}(\diamond x, \varepsilon) <_{\text{Pt}(\mathcal{U}(X))} \mathbf{b}_{\tilde{d}}(\alpha, \delta).$$

Applying the operation  $\text{Pt}(-)$  to  $\overline{i_X}$ , we obtain an isomorphism  $\text{Pt}(\overline{i_X}): \text{Pt}(\mathcal{U}(X)) \rightarrow \text{Pt}(\mathcal{U}(\text{Pt}(\mathcal{U}(X))))$ .

**Lemma 3.29.** *For each  $\alpha \in \text{Pt}(\mathcal{U}(X))$ , we have*

$$\text{Pt}(\overline{i_X})(\alpha) = \diamond \alpha.$$

*Proof.* First, suppose  $\mathbf{b}_{\tilde{d}}(\beta, \delta) \in \text{Pt}(\overline{i_X})(\alpha)$ . Then, there exists  $\mathbf{b}_d(x, \varepsilon) \in \alpha$  such that  $\mathbf{b}_{\tilde{d}}(\diamond x, \varepsilon) <_{\text{Pt}(\mathcal{U}(X))} \mathbf{b}_{\tilde{d}}(\beta, \delta)$ . Then,  $\tilde{d}(\beta, \alpha) \leq \tilde{d}(\beta, \diamond x) + \tilde{d}(\diamond x, \alpha) < \tilde{d}(\beta, \diamond x) + \varepsilon < \delta$ . Hence,  $\mathbf{b}_{\tilde{d}}(\beta, \delta) \in \diamond \alpha$ .

Conversely, let  $\mathbf{b}_{\tilde{d}}(\beta, \delta) \in \diamond \alpha$ . Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\tilde{d}(\beta, \alpha) + 2\theta < \delta$ . Since  $i_X: X \rightarrow \text{Pt}(\mathcal{U}(X))$  is dense, there exists  $x \in X$  such that  $\tilde{d}(\alpha, \diamond x) < \theta$  so that  $\mathbf{b}_d(x, \theta) \in \alpha$ . Moreover,  $\tilde{d}(\beta, \diamond x) + \theta \leq \tilde{d}(\beta, \alpha) + \tilde{d}(\alpha, \diamond x) + \theta < \tilde{d}(\beta, \alpha) + 2\theta < \delta$ . Thus,  $\mathbf{b}_{\tilde{d}}(\diamond x, \theta) <_{\text{Pt}(\mathcal{U}(X))} \mathbf{b}_{\tilde{d}}(\beta, \delta)$ , and hence  $\mathbf{b}_{\tilde{d}}(\beta, \delta) \in \text{Pt}(\overline{i_X})(\alpha)$ .  $\square$

By Proposition 3.23 and Proposition 3.27,  $(\text{Pt}(\mathcal{U}(X)), \widetilde{M})$  is a complete separated symmetric gus. Thus, we conclude as follows.

**Theorem 3.30.** *The isometry  $i_X: (X, M) \rightarrow (\text{Pt}(\mathcal{U}(X)), \widetilde{M})$  is a completion of  $(X, M)$ .*

**Remark 3.31.** By Theorem 3.19, the symmetric gus  $\text{Pt}(\mathcal{U}(X, M))$  embeds into the product  $\text{Pt}(\prod_{d \in M} \mathcal{M}(X, d)) \cong \prod_{d \in M} \text{Pt}(\mathcal{M}(X, d))$  of complete generalised uniform spaces as a closed subclass. This is one of the well-known construction of completions of uniform spaces [Kel75]. In particular, for a finite Dedekind symmetric gus  $(X, M)$ , the class  $\text{Pt}(\mathcal{U}(X, M))$  gives the usual completion of  $(X, M)$  with Cauchy filters.

The construction of  $\text{Pt}(\mathcal{U}(X, M))$ , however, is problematic from a predicative point of view because we have yet to prove that  $\text{Pt}(\mathcal{U}(X, M))$  is a set. If  $(X, M)$  is a finite Dedekind symmetric gus, this problem can be addressed by using the fact that for each pseudometric  $d \in M$ , the class  $\text{Pt}(\mathcal{M}(X, d))$  is isometric to the usual metric completion of  $(X, d)$  with Cauchy sequences [Pal07, Theorem 2.4] (this requires  $\text{AC}_\omega$ ). The completion of  $(X, M)$  is then obtained as a closed subset of the product  $\prod_{d \in M} \text{Pt}(\mathcal{M}(X, d))$ . Hence, the construction of the completion of Bishop's notion of uniform space [Bis67, Chapter 4, Problem 17] is predicative and unproblematic under the assumption of  $\text{AC}_\omega$ , which is a usual practice in Bishop constructive mathematics. See below for another way to cope with this predicativity issue.

**3.4.4. Predicativity issues.** We present another way to cope with the predicative issue raised in Remark 3.31. This method is specific to CZF, but it avoids  $\text{AC}_\omega$  and works for any symmetric gus. Note that to show that  $\text{Pt}(\mathcal{U}(X, M))$  forms a set, it suffices to show that for each symmetric gms  $(X, d)$ , the class  $\text{Pt}(\mathcal{M}(X, d))$  forms a set.

Recall that Fullness is the following statement, which is valid in CZF.

$$\forall X \forall Y \exists R [R \subseteq \text{mv}(X, Y) \ \& \ \forall s \in \text{mv}(X, Y) \exists r \in R \ r \subseteq s]$$

where  $\text{mv}(X, Y)$  is the *class* of total relations between sets  $X$  and  $Y$ .

Now, let  $(X, d)$  be a symmetric gms. By Fullness, there exists a subset

$$R \subseteq \mathbf{mv}(\mathbb{Q}^{>0}, U_{(X,d)})$$

such that for each  $s \in \mathbf{mv}(\mathbb{Q}^{>0}, U_{(X,d)})$  there exists  $r \in R$  such that  $r \subseteq s$ . Define a set

$$\overline{R} \stackrel{\text{def}}{=} \{r \in R \mid (\forall(\varepsilon, a) \in r) a \in \mathcal{C}_d^\varepsilon \text{ \& } (\forall(\varepsilon, a), (\delta, b) \in r) a_* \not\leq b_*\}.$$

The following is obvious.

**Lemma 3.32.** *For each  $r \in \overline{R}$ , the subset*

$$\alpha_r \stackrel{\text{def}}{=} \{a \in U_{(X,d)} \mid (\exists(\varepsilon, b) \in r) b <_X a\}$$

*is in  $\text{Pt}(\mathcal{M}(X, d))$ .*

Conversely, given  $\alpha \in \text{Pt}(\mathcal{M}(X, d))$ , put

$$r_\alpha \stackrel{\text{def}}{=} \{(\varepsilon, a) \in \mathbb{Q}^{>0} \times U_{(X,d)} \mid a \in \alpha \cap \mathcal{C}_d^\varepsilon\}.$$

By (U2), we have  $r_\alpha \in \mathbf{mv}(\mathbb{Q}^{>0}, U_{(X,d)})$ . Hence, by Fullness there exists  $r \in R$  such that  $r \subseteq r_\alpha$ . Clearly  $r \in \overline{R}$ , and  $\alpha_r \subseteq \alpha$ . Let  $a \in \alpha$ , and write  $a = \mathbf{b}_d(x, \varepsilon)$ . By (U1), there exists  $\delta < \varepsilon$  such that  $\mathbf{b}_d(x, \delta) \in \alpha$ . Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\delta + 2\theta < \varepsilon$ . By (U2), there exists  $\mathbf{b}_d(y, \theta) \in \alpha_r$ , and hence by (P2) applied to  $\alpha$ , we have  $d(x, y) < \delta + \theta$ . Then,  $d(x, y) + \theta < \delta + 2\theta < \varepsilon$ , and so  $\mathbf{b}_d(y, \theta) <_X \mathbf{b}_d(x, \varepsilon) = a$ . Then,  $a \in \alpha_r$  by (P3). Thus  $\alpha_r = \alpha$ .

**Proposition 3.33.** *The assignment  $r \mapsto \alpha_r$  is a surjection from  $\overline{R}$  to  $\text{Pt}(\mathcal{M}(X, d))$ . Hence,  $\text{Pt}(\mathcal{M}(X, d))$  is a set.*

The argument following Lemma 3.32 shows that every formal point of  $\text{Pt}(\mathcal{M}(X, d))$  is maximal. Hence, the above predicativity issue is a special case of the example discussed in [Pal06], [vdB13], and [AINS15]. Note that we do not require any extra axiom of CZF discussed in [vdB13] and [AINS15].

#### 4. FUNCTORIAL EMBEDDING OF LOCALLY COMPACT UNIFORM SPACES

In this section, we work with separated finite Dedekind symmetric gms's, which we simply call uniform spaces. This is the notion of uniform space treated in [Bis67]. We show that the category of locally compact uniform spaces can be embedded into that of overt locally compact completely regular formal topologies by extending the construction of localic completions to a full and faithful functor. The results in this section are rather straightforward generalisations of the corresponding results for metric spaces by Palmgren [Pal07, Section 4]. However, our characterisation of the cover of the localic completion of a locally compact uniform space (and thus also that of a locally compact metric space) may be interesting. We believe that our characterisation is an improvement over the previous one for metric spaces [Pal07, Theorem 4.17].

**4.1. Complete regularity.** Classically, the topology associated with a uniform space is completely regular. Hence, the localic completion of a uniform space should be (point-free) completely regular. Before making it precise, we recall the relevant notions from [Cur03].

Let  $\mathcal{S}$  be a formal topology. For a subset  $U \subseteq S$ , let  $U^* \stackrel{\text{def}}{=} \{b \in S \mid b \downarrow U \triangleleft \emptyset\}$ , and for any two subsets  $U, V \subseteq S$ , write  $U \lll V$  if  $S \triangleleft U^* \cup V$ . We write  $a \lll b$  for  $\{a\} \lll \{b\}$ . Put  $\mathbb{I} \stackrel{\text{def}}{=} \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\}$ . For subsets  $U, V \subseteq S$ , a *scale* from  $U$  to  $V$  is a family  $(U_q)_{q \in \mathbb{I}}$  of subsets of  $S$  such that  $U \triangleleft U_0$ ,  $U_1 \triangleleft V$  and  $U_p \lll U_q$  for each  $p, q \in \mathbb{I}$  such that  $p < q$ . We write  $U \ll V$  if there exists a scale from  $U$  to  $V$  and write  $a \ll b$  for  $\{a\} \ll \{b\}$ .

**Definition 4.1.** A formal topology  $\mathcal{S}$  is *completely regular* if it is equipped with a function  $\text{rc}: S \rightarrow \text{Pow}(S)$  such that

- (1)  $(\forall b \in \text{rc}(a)) \ b \ll a$ ,
- (2)  $a \triangleleft \text{rc}(a)$

for all  $a \in S$ .

**Lemma 4.2.** For any symmetric Dedekind gus  $(X, M)$ , we have

$$a <_X b \implies a \lll b$$

for all  $a, b \in U_X$ .

*Proof.* Let  $a, b \in U_X$ , and suppose that  $a <_X b$ . Write  $a = \mathbf{b}_d(x, \varepsilon)$ , and choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\mathbf{b}_d(x, \varepsilon + 3\theta) <_X b$ . Let  $c = \mathbf{b}_d(z, \theta) \in \mathcal{C}_d^\theta$ . Then, either  $d(x, z) > \varepsilon + \theta$  or  $d(x, z) < \varepsilon + 2\theta$ . In the former case, for any  $c' \in a \downarrow c$ , we have  $d(x, z) < \varepsilon + \theta$ , a contradiction. Thus,  $a \downarrow c \triangleleft_X \emptyset$  and so  $c \in a^*$ . In the latter case, we have

$$d(x, z) + \theta \leq \varepsilon + 3\theta,$$

so  $\mathbf{b}_d(z, \theta) \leq_X \mathbf{b}_d(x, \varepsilon + 3\theta) <_X b$ . Hence,  $U_X \triangleleft_X a^* \cup \{b\}$  by (U2). Therefore,  $a \lll b$ .  $\square$

**Proposition 4.3.** The localic completion of a symmetric Dedekind gus is completely regular.

*Proof.* Let  $(X, M)$  be a symmetric Dedekind gus. For each  $\mathbf{b}_d(x, \varepsilon) \in U_X$ , put

$$\text{rc}(\mathbf{b}_d(x, \varepsilon)) \stackrel{\text{def}}{=} \{\mathbf{b}_d(x, \delta) \in U_X \mid \delta < \varepsilon\}.$$

By (U1), we have  $a \triangleleft_X \text{rc}(a)$  for each  $a \in U_X$ . Let  $a, b \in U_X$  such that  $b \in \text{rc}(a)$ , and write  $a = \mathbf{b}_d(x, \varepsilon)$  and  $b = \mathbf{b}_d(x, \delta)$ . Since  $\delta < \varepsilon$ , there exists an order preserving bijection between  $[\delta, \varepsilon] \cap \mathbb{Q}$  and  $\mathbb{I}$ . Thus,  $b \lll a$  by Lemma 4.2, and so  $\mathcal{U}(X)$  is completely regular.  $\square$

**4.2. Compactness and local compactness.** The following notion of local compactness generalises the corresponding notion for metric spaces by Bishop [Bis67, Chapter 4, Definition 18].

**Definition 4.4.** A uniform space  $(X, M)$  is *totally bounded* if for each  $d \in M$ , the pseudo-metric space  $(X, d)$  is totally bounded, i.e.

$$(\forall \varepsilon \in \mathbb{Q}^{>0}) (\exists Y_\varepsilon \in \text{Fin}(X)) \ X \subseteq \bigcup_{y \in Y_\varepsilon} B_d(y, \varepsilon).$$

The set  $Y_\varepsilon$  is called an  $\varepsilon$ -net to  $X$  with respect to  $d$ . A uniform space is *compact* if it is complete and totally bounded. A uniform space  $X$  is *locally compact* if for each open ball  $B_d(x, \varepsilon)$  of  $X$ , there exists a compact subset  $K \subseteq X$  such that  $B_d(x, \varepsilon) \subseteq K$ . Thus, every compact uniform space is locally compact.

**Remark 4.5.** The above notion of local compactness is not invariant under homeomorphisms, which is often considered unsatisfactory. For example, the real line is locally compact in the above sense, but  $(0, 1)$  is not. In this paper, we are not aiming at a better alternative to Bishop's definition of local compactness but one that is compatible with it.

Compactness in formal topology is defined by the covering compactness.

**Definition 4.6.** A formal topology  $\mathcal{S}$  is *compact* if

$$S \triangleleft U \implies (\exists U_0 \in \text{Fin}(U)) S \triangleleft U_0$$

for all  $U \subseteq S$ .

**Definition 4.7.** Let  $\mathcal{S}$  be a formal topology. For each  $a, b \in S$  define

$$a \ll b \stackrel{\text{def}}{\iff} (\forall U \in \text{Pow}(S)) [b \triangleleft U \implies (\exists U_0 \in \text{Fin}(U)) a \triangleleft U_0].$$

A formal topology  $\mathcal{S}$  is *locally compact* if it is equipped with a function  $\text{wb}: S \rightarrow \text{Pow}(S)$  such that

- (1)  $(\forall b \in \text{wb}(a)) b \ll a$ ,
- (2)  $a \triangleleft \text{wb}(a)$

for all  $a \in S$ . Since the relation  $\ll$  is a proper class in general, the function  $\text{wb}: S \rightarrow \text{Pow}(S)$  is an essential part of the definition of local compactness.

Given a uniform space  $(X, M)$ , define a relation  $\sqsubseteq$  on  $\text{Pow}(U_X)$  by

$$U \sqsubseteq V \stackrel{\text{def}}{\iff} (\exists (d, \varepsilon) \in \text{Rad}(M)) U \downarrow \mathcal{C}_d^\varepsilon \leq_X V$$

for all  $U, V \subseteq U_X$ . By (U2), we have

$$U \sqsubseteq V \implies U \triangleleft_X V.$$

**Lemma 4.8.** Let  $(X, M)$  be a uniform space. Then, the following are equivalent for all  $a \in U_X$  and  $U \subseteq U_X$ :

- (1)  $(\exists V \in \text{Fin}(U_X)) a_* \subseteq V_* \text{ \& } V <_X U$ ;
- (2)  $(\exists V \in \text{Fin}(U_X)) a \sqsubseteq V <_X U$ ;
- (3)  $(\exists V \in \text{Fin}(U_X)) a \triangleleft_X V <_X U$ .

*Proof.* (1)  $\implies$  (2) Suppose that (1) holds. Then, there exists  $V \in \text{Fin}(U_X)$  such that  $a_* \subseteq V_*$  and  $V <_X U$ . Write  $V = \{\mathbf{b}_{d_0}(x_0, \varepsilon_0), \dots, \mathbf{b}_{d_n}(x_n, \varepsilon_n)\}$ , and choose  $\theta \in \mathbb{Q}^{>0}$  such that  $V' \stackrel{\text{def}}{=} \{\mathbf{b}_{d_i}(x_i, \varepsilon_i + \theta) \mid i \leq n\} <_X U$ . Let  $d \stackrel{\text{def}}{=} \sup \{d_i \mid i \leq n\}$ , and let  $\mathbf{b}_\rho(y, \delta) \in a \downarrow \mathcal{C}_d^\theta$ . Then, there exists  $i \leq n$  such that  $d_i(x_i, y) < \varepsilon_i$ , so we have  $d_i(x_i, y) + \delta < \varepsilon_i + \delta \leq \varepsilon_i + \theta$ . Thus,  $\mathbf{b}_\rho(y, \delta) <_X \mathbf{b}_{d_i}(x_i, \varepsilon_i + \theta)$ , and hence  $a \sqsubseteq V'$ .

(2)  $\implies$  (3) We have  $a \sqsubseteq V \implies a \triangleleft_X V$ .

(3)  $\implies$  (1) We have  $a \triangleleft_X V \implies a_* \subseteq V_*$ . □

The cover of the localic completion of a locally compact uniform space admits an elementary characterisation.

**Proposition 4.9.** Let  $(X, M)$  be a locally compact uniform space. Then, the following are equivalent for all  $a \in U_X$  and  $U \subseteq U_X$ :

- (1)  $a \triangleleft_X U$ ;
- (2)  $(\forall b <_X a) (\exists V \in \text{Fin}(U_X)) b_* \subseteq V_* \text{ \& } V <_X U$ ;
- (3)  $(\forall b <_X a) (\exists V \in \text{Fin}(U_X)) b \sqsubseteq V <_X U$ ;

$$(4) (\forall b <_X a) (\exists V \in \text{Fin}(U_X)) b \triangleleft_X V <_X U.$$

*Proof.* By Lemma 4.8, it suffices to show that (1) implies (2). Given  $U \subseteq U_X$ , define a predicate  $\Phi_U$  on  $U_X$  by

$$\Phi_U(a) \stackrel{\text{def}}{\iff} (\forall b <_X a) (\exists V \in \text{Fin}(U_X)) b_* \subseteq V_* \ \& \ V <_X U.$$

We show that

$$a \triangleleft_X U \implies \Phi_U(a)$$

for all  $a \in U_X$  by induction on  $\triangleleft_X$ . We must check the conditions (ID1) – (ID3) for the localised axiom-set consisting of (U1) and (U2').

The conditions (ID1) and (ID2) are straightforward to check, using Lemma 3.10. For (ID3), we have two axioms to be checked.

(U1)  $\frac{(\forall b <_X a) \Phi_U(b)}{\Phi_U(a)}$ : Suppose that  $\Phi_U(b)$  for all  $b <_X a$ . Let  $b <_X a$ . Then, there

exists  $c \in U_X$  such that  $b <_X c <_X a$ . Since  $\Phi_U(c)$ , there exists  $V \in \text{Fin}(U_X)$  such that  $b_* \subseteq V_*$  and  $V <_X U$ . Hence,  $\Phi_U(a)$ .

(U2')  $\frac{(\forall c \in \mathcal{C}_\rho^\theta \downarrow a) \Phi_U(c)}{\Phi_U(a)}$  for each  $(\rho, \theta) \in \text{Rad}(M)$ : Suppose that  $\Phi_U(c)$  for all  $c \in$

$\mathcal{C}_\rho^\theta \downarrow a$ . Let  $b <_X a$ , and write  $a = \mathbf{b}_d(x, \varepsilon)$  and  $b = \mathbf{b}_{d'}(y, \delta)$ . Since  $(X, M)$  is locally compact, there exists a compact subset  $K \subseteq X$  such that  $\mathbf{B}_d(x, \varepsilon) \subseteq K$ . Choose  $\xi \in \mathbb{Q}^{>0}$  such that  $2\xi < \theta$  and  $d(x, y) + \delta + 4\xi < \varepsilon$ . Let  $Z \stackrel{\text{def}}{=} \{z_0, \dots, z_{n-1}\}$  be a  $\xi$ -net to  $K$  with respect to  $\rho' \stackrel{\text{def}}{=} \sup\{d, \rho\}$ . Split  $Z$  into two finitely enumerable subsets  $Z^+$  and  $Z^-$  such that  $Z = Z^+ \cup Z^-$  and

- $z \in Z^+ \implies d(z, x) < \varepsilon - 2\xi$ ,
- $z \in Z^- \implies d(z, x) > \varepsilon - 3\xi$ .

Let  $z \in Z^+$ . Since  $\mathbf{b}_{\rho'}(z, 2\xi) \in \mathcal{C}_\rho^\theta \downarrow a$ , we have  $\Phi_U(\mathbf{b}_{\rho'}(z, 2\xi))$ . Hence there exists  $V_z \in \text{Fin}(U_X)$  such that  $\mathbf{B}_{\rho'}(z, \xi) \subseteq V_{z*}$  and  $V_z <_X U$ . Since  $Z^+$  is finitely enumerable, there exists  $V \in \text{Fin}(U_X)$  such that  $\bigcup_{z \in Z^+} \mathbf{B}_{\rho'}(z, \xi) \subseteq V_*$  and  $V <_X U$ . Now, it suffices to show that  $b_* \subseteq \bigcup_{z \in Z^+} \mathbf{B}_{\rho'}(z, \xi)$ . Let  $y' \in b_*$ . Then, there exists  $i < n$  such that  $\rho'(y', z_i) < \xi$ . Then

$$\begin{aligned} d(z_i, x) &\leq d(z_i, y') + d(y', y) + d(y, x) \\ &< \xi + \delta + d(y, x) < \varepsilon - 3\xi, \end{aligned}$$

and thus  $z_i \in Z^+$ . Hence,  $y' \in \bigcup_{z \in Z^+} \mathbf{B}_{\rho'}(z, \xi)$ , and therefore  $\Phi_U(a)$ .  $\square$

**Remark 4.10.** By Proposition 4.9, inductive generation of the cover of the localic completion of a locally compact uniform space does not require the Regular Extension Axiom.

**Example 4.11** ([Pal07, Example 3.3]). Consider the real plane  $X \stackrel{\text{def}}{=} \mathbb{R}^2$ , which is a locally compact uniform space (it is even a metric space). Let  $x = (0, 0)$ ,  $y = (-4, 0)$ , and  $z = (4, 0)$ . Put  $a = \mathbf{b}_d(x, 3)$ ,  $b = \mathbf{b}_d(y, 5)$ , and  $c = \mathbf{b}_d(z, 5)$ , where  $d$  is the standard distance on  $\mathbb{R}^2$  (see the left figure in Figure 1). Then,  $a \triangleleft_X \{b, c\}$ . This can be seen as follows: if we shrink the radius of  $a$  by  $\varepsilon \in \mathbb{Q}^{>0}$  and let  $a' = \mathbf{b}_d(x, 3 - \varepsilon)$ , we can find a sufficiently small  $\delta \in \mathbb{Q}^{>0}$  such that  $a' \downarrow \mathcal{C}_d^\delta \leq_X \{b, c\}$  as can be visually seen from the right figure in Figure 1. Note that we should not conclude  $a \triangleleft_X \{b, c\}$  from the left figure just because  $a_* \subseteq b_* \cup c_*$ . This relies on the spatiality of  $\mathcal{U}(\mathbb{R}^2)$ .

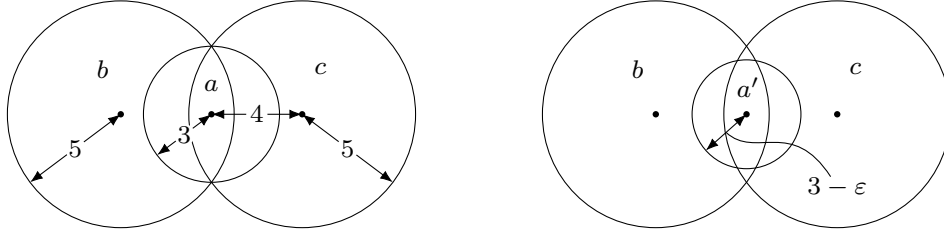


Figure 1: Example 4.11

**Example 4.12.** The characterisation of the cover in Proposition 4.9 can be considered as a natural generalisation of the characterisation of the cover of the localic reals by Johnstone [Joh82, Chapter IV, Section 1.1, Lemma]. We restate his characterisation in terms of formal reals  $\mathcal{R}$ , which is shown to be identical to the localic completion of the space of rational numbers [Pal07, Example 2.2]. Recall that  $\mathcal{R}$  is a formal topology with a base  $S_{\mathcal{R}} = \{(p, q) \in \mathbb{Q} \times \mathbb{Q} \mid p < q\}$  ordered by  $(p, q) \leq_{\mathcal{R}} (r, s) \stackrel{\text{def}}{\iff} r \leq p \text{ \& } q \leq s$ . For each  $(p, q), (r, s) \in S_{\mathcal{R}}$  define  $(p, q) <_{\mathcal{R}} (r, s) \stackrel{\text{def}}{\iff} r < p < q < s$ . The axioms of  $\mathcal{R}$  are the following:

- (R1)  $(p, q) \triangleleft_{\mathcal{R}} \{(r, s) \in S_{\mathcal{R}} \mid (r, s) <_{\mathcal{R}} (p, q)\}$ ,
- (R2)  $(p, q) \triangleleft_{\mathcal{R}} \{(p, s), (r, q)\}$  for each  $p < r < s < q$ .

Then, for each  $U \subseteq S_{\mathcal{R}}$  and  $(p, q) \in S_{\mathcal{R}}$ , we have

$$(p, q) \triangleleft_{\mathcal{R}} U \iff \forall (p', q') <_{\mathcal{R}} (p, q) \exists (p_0, q_0), \dots, (p_n, q_n) \in \downarrow U \\ p' = p_0 \text{ \& } q_n = q' \text{ \& } (\forall i < n) p_i \leq p_{i+1} < q_i \leq q_{i+1},$$

where  $\downarrow U$  is the downward closure of  $U$  with respect to  $\leq_{\mathcal{R}}$ . The proof is by a straightforward induction on  $\triangleleft_{\mathcal{R}}$ .

**Corollary 4.13.** *For any locally compact uniform space  $X$ , we have*

$$a <_X b \implies a \ll b$$

for all  $a, b \in U_X$ .

By (U1), we obtain the following.

**Theorem 4.14.** *The localic completion of a locally compact uniform space is locally compact.*

**Theorem 4.15.** *A uniform space  $(X, M)$  is totally bounded if and only if  $\mathcal{U}(X)$  is compact.*

*Proof.* Suppose that  $(X, M)$  is totally bounded. Let  $U \subseteq U_X$ , and suppose that  $U_X \triangleleft_X U$ . Choose any  $d \in M$  and  $\varepsilon \in \mathbb{Q}^{>0}$ , and let  $\{x_0, \dots, x_{n-1}\}$  be an  $\varepsilon$ -net to  $X$  with respect to  $d$ . By (U2), we have  $U_X \triangleleft_X \mathcal{C}_d^\varepsilon \triangleleft_X \{\mathbf{b}_d(x_i, 2\varepsilon) \mid i < n\}$ . Thus, there exists  $\mathbf{b}_d(y, \delta) \in U_X$  such that  $U_X \triangleleft_X \mathbf{b}_d(y, \delta)$ . Since  $\mathbf{b}_d(y, 2\delta) \triangleleft_X U$ , there exists  $V \in \text{Fin}(U_X)$  such that  $\mathbf{b}_d(y, \delta) \triangleleft_X V <_X U$  by Proposition 4.9. Then, there exists  $U_0 \in \text{Fin}(U)$  such that  $U_X \triangleleft_X U_0$ . Therefore,  $\mathcal{U}(X)$  is compact.

Conversely, suppose that  $\mathcal{U}(X)$  is compact. Let  $d \in M$  and  $\varepsilon \in \mathbb{Q}^{>0}$ . Since  $U_X \triangleleft_X \mathcal{C}_d^\varepsilon$ , there exists  $V = \{\mathbf{b}_d(x_0, \varepsilon), \dots, \mathbf{b}_d(x_{n-1}, \varepsilon)\} \in \text{Fin}(\mathcal{C}_d^\varepsilon)$  such that  $U_X \triangleleft_X V$ . Then,  $X = U_{X*} \subseteq V_* = \bigcup_{i < n} B_d(x_i, \varepsilon)$ , and hence  $\{x_0, \dots, x_{n-1}\}$  is an  $\varepsilon$ -net to  $X$  with respect to  $d$ . Therefore,  $X$  is totally bounded.  $\square$

### 4.3. Functorial embedding.

**Definition 4.16.** A function  $f: X \rightarrow Y$  from a locally compact uniform space  $(X, M)$  to a uniform space  $(Y, N)$  is *continuous* if  $f$  is uniformly continuous on each open ball of  $X$ , i.e. for each  $x \in X$ ,  $d \in M$  and  $\varepsilon \in \mathbb{Q}^{>0}$ ,

$$(\forall \rho \in N) (\forall \delta \in \mathbb{Q}^{>0}) (\exists d' \in M) (\exists \varepsilon' \in \mathbb{Q}^{>0}) \\ [(\forall x_1, x_2 \in B_d(x, \varepsilon)) d'(x_1, x_2) < \varepsilon' \implies \rho(f(x_1), f(x_2)) < \delta].$$

Since the image of a totally bounded uniform space under a uniformly continuous function is again totally bounded, continuous functions between locally compact uniform spaces are closed under composition. Thus, the locally compact uniform spaces and continuous functions form a category, which we denote by **LKUSpa**.

**Lemma 4.17.** *A locally compact uniform space is complete.*

*Proof.* Let  $(X, M)$  be a locally compact uniform space. Let  $\mathcal{F}$  be a Cauchy filter on  $X$ . Choose any  $d \in M$  and  $\varepsilon \in \mathbb{Q}^{>0}$ . By (CF4), there exists  $x \in X$  such that  $B_d(x, \varepsilon) \in \mathcal{F}$ . Since  $X$  is locally compact, there exists a compact subset  $K \subseteq X$  such that  $B_d(x, \varepsilon) \subseteq K$ . Let  $\mathcal{G} = \{U \in \mathcal{F} \mid U \subseteq K\}$ . It is easy to see that  $\mathcal{G}$  is a Cauchy filter on  $K$ . Since  $K$  is complete,  $\mathcal{G}$  converges to some  $z \in K$ . Since  $\mathcal{G} \subseteq \mathcal{F}$ ,  $\mathcal{F}$  also converges to  $z$ .  $\square$

Thus, for each locally compact uniform space  $X$ , the embedding  $i_X: X \rightarrow \text{Pt}(\mathcal{U}(X))$  defined by (3.4) is a uniform isomorphism.

Given any function  $f: X \rightarrow Y$  between uniform spaces  $X$  and  $Y$ , define a relation  $r_f \subseteq U_X \times U_Y$  by

$$a r_f b \stackrel{\text{def}}{\iff} (\exists b' <_Y b) f[a_*] \subseteq b'_* \quad (4.1)$$

for each  $a \in U_X$  and  $b \in U_Y$ . The relation  $r_f$  is a natural generalisation of the relation  $D_f$  defined by Palmgren in the setting of metric spaces [Pal07, Section 5].

The following lemma corresponds to [Pal07, Theorem 5.2]. Note that the assumption in [Pal07, Theorem 5.2] that  $X$  is locally compact is not necessary.

**Lemma 4.18.** *If  $f: (X, M) \rightarrow (Y, N)$  is uniformly continuous on each open ball of  $X$ , then  $r_f$  is a formal topology map from  $\mathcal{U}(X)$  to  $\mathcal{U}(Y)$ .*

*Proof.* We check (FTM1), (FTM2), (FTM3a), and (FTM3b).

(FTM1) Let  $a \in U_X$ . Choose any  $d \in N$  and  $\varepsilon \in \mathbb{Q}^{>0}$ . Since  $f$  is uniformly continuous on  $a_*$ , there exist  $\rho \in M$  and  $\delta \in \mathbb{Q}^{>0}$  such that

$$(\forall x, x' \in a_*) \rho(x, x') < \delta \implies d(f(x), f(x')) < \varepsilon.$$

Then by (U2), we have  $a \triangleleft_X a \downarrow \mathcal{C}_\rho^\delta \subseteq r_f^- \mathcal{C}_d^{2\varepsilon} \subseteq r_f^- U_Y$ .

(FTM2) Let  $b, c \in U_Y$  and  $a \in r_f^- b \downarrow r_f^- c$ . Then, there exist  $b' <_Y b$  and  $c' <_Y c$  such that  $f[a_*] \subseteq b'_* \cap c'_*$ . Write  $b' = b_{d_1}(y, \delta)$  and  $c' = b_{d_2}(z, \xi)$ , and put  $d = \sup\{d_1, d_2\}$ . Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $b_{d_1}(y, \delta + 2\theta) <_Y b$  and  $b_{d_2}(z, \xi + 2\theta) <_Y c$ . Since  $f$  is uniformly continuous on  $a_*$ , there exist  $\rho \in M$  and  $\varepsilon \in \mathbb{Q}^{>0}$  such that

$$(\forall x, x' \in a_*) \rho(x, x') < \varepsilon \implies d(f(x), f(x')) < \theta.$$

Let  $b_{\rho'}(x', \varepsilon') \in a \downarrow \mathcal{C}_\rho^\varepsilon$ . Then,  $f[b_{\rho'}(x', \varepsilon')_*] \subseteq b_d(f(x'), \theta)_*$ . Since  $b_d(f(x'), 2\theta) \in b_{d_1}(y, \delta + 2\theta) \downarrow b_{d_2}(z, \xi + 2\theta) \subseteq b \downarrow c$ , we have  $b_{\rho'}(x', \varepsilon') \in r_f^-(b \downarrow c)$ . Hence by (U2), we have  $a \triangleleft_X r_f^-(b \downarrow c)$ .

(FTM3a) Obvious.



(FTM3b) For (U1), we have  $r_f^{-1}b \triangleleft_X r_f^{-1}\{b' \in U_Y \mid b' <_Y b\}$  for all  $b \in U_Y$  by Lemma 3.10 (2). For (U2), the argument is similar to the proof of the case (FTM1).  $\square$

**Lemma 4.19.** *Let  $X$  be a locally compact uniform space, and let  $Y$  be a complete uniform space. For any formal topology map  $r: \mathcal{U}(X) \rightarrow \mathcal{U}(Y)$ , the composition*

$$f = i_Y^{-1} \circ \text{Pt}(r) \circ i_X$$

*is uniformly continuous on each open ball of  $X$ .*

*Proof.* See [Pal07, Theorem 5.1].  $\square$

**Lemma 4.20.** *Let  $X$  and  $Y$  be complete uniform spaces, and let  $f: X \rightarrow Y$  be a function which is uniformly continuous on each open ball of  $X$ . Then, the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{i_X} & \text{Pt}(\mathcal{U}(X)) \\ f \downarrow & & \downarrow \text{Pt}(r_f) \\ Y & \xleftarrow{i_Y^{-1}} & \text{Pt}(\mathcal{U}(Y)) \end{array}$$

*Proof.* See [Pal07, Lemma 5.7 (i)].  $\square$

**Lemma 4.21.** *Let  $X$  be a locally compact uniform space, and let  $Y$  be a complete uniform space. Then, for any formal topology map  $r: \mathcal{U}(X) \rightarrow \mathcal{U}(Y)$ , we have  $r_f = r$ , where  $f \stackrel{\text{def}}{=} i_Y^{-1} \circ \text{Pt}(r) \circ i_X$ .*

*Proof.* See [Pal07, Lemma 5.7 (ii)].  $\square$

**Lemma 4.22.** *Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  be continuous functions between locally compact uniform spaces. Then,  $r_{g \circ f} = r_g \circ r_f$ .*

*Proof.* See the proof of [Pal07, Theorem 5.8].  $\square$

Similarly, we can show that  $r_{\text{id}_X} = \text{id}_{\mathcal{U}(X)}$  for any locally compact uniform space  $X$ . Hence, we conclude as follows.

**Theorem 4.23.** *The localic completion  $\mathcal{U}$  extends to a full and faithful functor*

$$\mathcal{U}: \mathbf{LKUSpa} \rightarrow \mathbf{OLKCRg}$$

*from the category of locally compact uniform spaces  $\mathbf{LKUSpa}$  to that of overt locally compact completely regular formal topologies  $\mathbf{OLKCRg}$ .*

*Proof.* For each morphism  $f: X \rightarrow Y$  of  $\mathbf{LKUSpa}$ , define  $\mathcal{U}(f) \stackrel{\text{def}}{=} r_f$ . Then, by Lemma 4.22,  $\mathcal{U}$  is a functor. By Lemma 4.20,  $\mathcal{U}$  is faithful, and by Lemma 4.19 and Lemma 4.21,  $\mathcal{U}$  is full.  $\square$

By an abuse of terminology, we call the functor  $\mathcal{U}: \mathbf{LKUSpa} \rightarrow \mathbf{OLKCRg}$  the localic completion of locally compact uniform spaces.

**4.4. Preservation of products.** Palmgren [Pal07] showed that the localic completion of locally compact metric spaces preserve finite products. We extend his result to the setting of uniform spaces.

**Lemma 4.24.** *A binary product of locally compact uniform spaces (as generalised uniform spaces) is locally compact.*

*Proof.* The proof is analogous to the metric case [Bis67, Chapter 4, Proposition 12].  $\square$

**Lemma 4.25.** *Let  $(X, M)$  and  $(Y, N)$  be generalised uniform spaces, and let  $f: X \rightarrow Y$  be a homomorphism. Let  $r$  and  $s$  be the relations between  $\mathcal{U}(X)$  and  $\mathcal{U}(Y)$  as given by (4.1) and (3.3) respectively. Then,  $r$  and  $s$  are equal as formal topology maps.*

*Proof.* We must show that  $\mathcal{A}r^- \mathbf{b}_\rho(y, \delta) = \mathcal{A}s^- \mathbf{b}_\rho(y, \delta)$  for each  $\mathbf{b}_\rho(y, \delta) \in U_Y$ .

First, suppose that  $\mathbf{b}_d(x, \varepsilon) r \mathbf{b}_\rho(y, \delta)$ . Then, there exists  $\mathbf{b}_{\rho'}(y', \delta') <_Y \mathbf{b}_\rho(y, \delta)$  such that  $f[\mathbf{B}_d(x, \varepsilon)] \subseteq \mathbf{B}_{\rho'}(y', \delta')$ . Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\rho(y, y') + \delta' + \theta < \delta$ , and let  $d' \in M$  such that  $d' \omega_f \rho'$ . By (U2), we have

$$\mathbf{b}_d(x, \varepsilon) \triangleleft_X \mathbf{b}_d(x, \varepsilon) \downarrow \mathcal{C}_{d'}^\theta.$$

Let  $\mathbf{b}_{d^*}(x', \varepsilon') \in \mathbf{b}_d(x, \varepsilon) \downarrow \mathcal{C}_{d'}^\theta$ . Then,  $d^* \omega_f \rho$ , and we have

$$\rho(y, f(x')) + \varepsilon' \leq \rho(y, y') + \rho'(y', f(x')) + \varepsilon' < \rho(y, y') + \delta' + \varepsilon' < \delta.$$

Thus,  $\mathbf{b}_\rho(f(x'), \varepsilon') <_Y \mathbf{b}_\rho(y, \delta)$ , so  $\mathbf{b}_d(x, \varepsilon) \triangleleft_X s^- \mathbf{b}_\rho(y, \delta)$ .

Next, suppose that  $\mathbf{b}_d(x, \varepsilon) s \mathbf{b}_\rho(y, \delta)$ . Then,

$$\rho(f(x), f(x')) \leq d(x, x')$$

for any  $x' \in X$ , so  $f[\mathbf{B}_d(x, \varepsilon)] \subseteq \mathbf{B}_\rho(f(x), \varepsilon)$ . Thus,  $\mathbf{b}_d(x, \varepsilon) r \mathbf{b}_\rho(y, \delta)$ .  $\square$

Since the projections from  $X \times Y$  to  $X$  and  $Y$  are homomorphisms, we have the following by Proposition 3.14.

**Theorem 4.26.** *For any locally compact uniform spaces  $(X, M)$  and  $(Y, N)$ , we have*

$$\mathcal{U}(X) \times \mathcal{U}(Y) \cong \mathcal{U}(X \times Y).$$

Similarly, by Theorem 3.15, the localic completion preserves inhabited countable products of compact uniform spaces. Note that inhabited countable products of compact uniform spaces are again compact since countable products of complete uniform spaces are complete and inhabited countable products of total bounded uniform spaces are total bounded (see [BB85, Chapter 4, Problems 26]).

**Example 4.27.** The Hilbert cube  $\prod_{n \in \mathbb{N}} [0, 1]$  is an inhabited countable product of the unit interval  $[0, 1]$ , which is a compact metric space. The localic completion of  $[0, 1]$  is the formal unit interval  $\mathcal{I}[0, 1]$ , which is the overt weakly closed subtopology of the formal real  $\mathcal{R}$  (cf. Example 4.12) determined by the splitting subset

$$\text{Pos}_{\mathcal{I}[0,1]} \stackrel{\text{def}}{=} \{(p, q) \in S_{\mathcal{R}} \mid p < 1 \ \& \ 0 < q\}.$$

Note that  $\mathcal{I}[0, 1]$  is obtained as the localic completion of the rational interval  $[0, 1] \cap \mathbb{Q}$  since the unit interval arises as its metric completion (cf. Proposition 3.28). In this case, we have  $\mathcal{U}(\prod_{n \in \mathbb{N}} [0, 1]) \cong \prod_{n \in \mathbb{N}} \mathcal{I}[0, 1]$ .

## 5. CONNECTION TO POINT-FREE COMPLETION

In [28], Fox introduced the notion of a uniform formal topology, a formal topology equipped with a covering uniformly, and established an adjunction between the category of uniform spaces equipped with covering uniformities and that of uniform formal topologies. He also defined the completion of a uniform formal topology, which is classically equivalent to the completion of a uniform locale by Kříž [Kříž86].

The construction given in this section on symmetric gus's is equivalent to applying the left adjoint of the adjunction defined by Fox followed by the completion of uniform formal topologies. We show that this construction is equivalent to the localic completion.

First, we recall the relevant notions from [Fox05, Chapter 6, Section 2]. Our presentation is slightly different, but equivalent to the one given in [Fox05].

**Definition 5.1.** Let  $\mathcal{S}$  be an overt formal topology with positivity  $\text{Pos}$ . A *cover* of  $\mathcal{S}$  is a subset  $C \subseteq S$  such that  $S \triangleleft C$ . For covers  $C, C' \in \text{Pow}(S)$  of  $\mathcal{S}$ , define

$$\begin{aligned} C \leq C' &\stackrel{\text{def}}{\iff} (\forall a \in C) (\exists a' \in C') a \triangleleft a', \\ C <^* C' &\stackrel{\text{def}}{\iff} (\forall a \in C) (\exists a' \in C') St_C(a) \triangleleft a', \end{aligned}$$

where  $St_C(a) \stackrel{\text{def}}{=} \{b \in C \mid \text{Pos}(b \downarrow a)\}$ . A *uniformity* on an overt formal topology  $\mathcal{S}$  is a set  $\mathcal{C}$  of covers of  $\mathcal{S}$  such that

- (1)  $(\forall C_1, C_2 \in \mathcal{C}) (\exists C_3 \in \mathcal{C}) C_3 \leq C_1 \ \& \ C_3 \leq C_2$ ,
- (2)  $(\forall C \in \mathcal{C}) (\exists C' \in \mathcal{C}) C' <^* C$ ,
- (3)  $(\forall a \in S) a \triangleleft \mathbf{uc}(a)$ ,

where  $\mathbf{uc}(a) \stackrel{\text{def}}{=} \{b \in S \mid (\exists C \in \mathcal{C}) St_C(b) \triangleleft a\}$ .

A *uniform formal topology* is a pair  $(\mathcal{S}, \mathcal{C})$  where  $\mathcal{S}$  is an overt formal topology and  $\mathcal{C}$  is a uniformity on  $\mathcal{S}$ .

**Definition 5.2.** Let  $(\mathcal{S}, \mathcal{C})$  be a uniform formal topology. Define a preorder  $\leq$  on  $S$  by  $a \leq b \stackrel{\text{def}}{\iff} a \triangleleft b$ . The *completion* of  $\mathcal{S}$  is a formal topology  $\overline{\mathcal{S}} = (S, \bar{\triangleleft}, \leq)$  inductively generated by the following axioms.

- (CP1)  $a \bar{\triangleleft} \mathbf{uc}(a)$ ;
- (CP2)  $a \bar{\triangleleft} C$  for each  $C \in \mathcal{C}$ ;
- (CP3)  $a \bar{\triangleleft} \{a \mid \text{Pos}(a)\}$

where  $\text{Pos}$  is the positivity of  $\mathcal{S}$ .

A symmetric gus  $(X, M)$  determines a uniform formal topology  $(\mathcal{S}_X, \mathcal{C}_M)$ : the formal topology  $\mathcal{S}_X = (U_X, \triangleleft_X^*, \preceq_X)$  is the usual topology induced by the uniformity  $M$ , i.e.

$$\begin{aligned} U_X &\stackrel{\text{def}}{=} \text{Rad}(M) \times X, \\ \mathbf{b}_d(x, \varepsilon) \preceq_X \mathbf{b}_\rho(y, \delta) &\stackrel{\text{def}}{\iff} \mathbf{b}_d(x, \varepsilon)_* \subseteq \mathbf{b}_\rho(y, \delta)_* \iff B_d(x, \varepsilon) \subseteq B_\rho(y, \delta), \\ \mathbf{b}_d(x, \varepsilon) \triangleleft_X^* U &\stackrel{\text{def}}{\iff} \mathbf{b}_d(x, \varepsilon)_* \subseteq U_*, \end{aligned}$$

where we use the same notation for the elements of  $U_X$  adopted in Section 3.2. Note that the positivity of  $\mathcal{S}_X$  is  $U_X$ . The uniformity  $\mathcal{C}_M$  is given by

$$\mathcal{C}_M \stackrel{\text{def}}{=} \{\mathcal{C}_d^\varepsilon \mid (d, \varepsilon) \in \text{Rad}(M)\},$$

where  $\mathcal{C}_d^\varepsilon$  is defined by (3.2). The pair  $(\mathcal{S}_X, \mathcal{C}_M)$  is the standard uniform formal topology with the usual topology induced by the uniformity  $M$ . The completion of  $(\mathcal{S}_X, \mathcal{C}_M)$  is an inductively generated formal topology  $\overline{\mathcal{S}}_X = (U_X, \blacktriangleleft_X, \preceq_X)$ , where  $(U_X, \preceq_X)$  is the underlying preorder of  $\mathcal{S}_X$  and the cover  $\blacktriangleleft_X$  is inductively generated by the following axioms:

- (C1)  $a \blacktriangleleft_X \{b \in U_X \mid b \prec_X a\}$ ;
- (C2)  $a \blacktriangleleft_X \mathcal{C}_d^\varepsilon$  for each  $(d, \varepsilon) \in \text{Rad}(M)$ ,

for each  $a \in U_X$ , where

$$a \prec_X b \stackrel{\text{def}}{\iff} (\exists (d, \varepsilon) \in \text{Rad}(M)) (\forall c \in \mathcal{C}_d^\varepsilon) c_* \checkmark a_* \rightarrow c \preceq_X b.$$

The axioms (C2) are equivalent to the following axioms:

$$(C2') \quad a \blacktriangleleft_X \mathcal{C}_d^\varepsilon \downarrow_{\preceq_X} a \text{ for each } (d, \varepsilon) \in \text{Rad}(M),$$

where  $\downarrow_{\preceq_X}$  is defined with respect to  $\preceq_X$ . Thus, the axioms (C1) and (C2') form a localised axiom-set on  $U_X$  with respect to  $\preceq_X$ .

For each  $(d, \varepsilon) \in \text{Rad}(M)$ , define a relation  $\prec_X^{(d, \varepsilon)}$  on  $U_X$  by

$$a \prec_X^{(d, \varepsilon)} b \stackrel{\text{def}}{\iff} (\forall c \in \mathcal{C}_d^\varepsilon) c_* \checkmark a_* \rightarrow c \preceq_X b.$$

Note that  $a \prec_X b \iff (\exists (d, \varepsilon) \in \text{Rad}(M)) a \prec_X^{(d, \varepsilon)} b$ .

**Lemma 5.3.** *Let  $(X, M)$  be a symmetric gus. Then,*

- (1)  $a \leq_X b \implies a \preceq_X b$ ,
- (2)  $a <_X b \implies a \prec_X b$ ,

for all  $a, b \in U_X$ .

*Proof.* (1) This is equivalent to Lemma 3.10 (3).

(2) Let  $a, b \in U_X$ , and suppose that  $a <_X b$ . Write  $a = \mathbf{b}_d(x, \varepsilon)$  and  $b = \mathbf{b}_\rho(y, \delta)$ , and choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\rho(y, x) + \varepsilon + 2\theta < \delta$ . We show that  $a \prec_X^{(d, \theta)} b$ . Let  $c = \mathbf{b}_d(z, \theta) \in \mathcal{C}_d^\theta$ , and suppose that  $c_* \checkmark a_*$ . Then,  $d(x, z) < \varepsilon + \theta$ , so

$$\begin{aligned} \rho(y, z) + \theta &\leq \rho(y, x) + \rho(x, z) + \theta \\ &\leq \rho(y, x) + d(x, z) + \theta \\ &< \rho(y, x) + \varepsilon + 2\theta < \delta. \end{aligned}$$

Thus,  $c \leq_X b$ , so that  $c \preceq_X b$ . Hence  $a \prec_X^{(d, \theta)} b$ , and therefore  $a \prec_X b$ .  $\square$

The following is a corollary of Lemma 4.8, which holds for any symmetric gus as well.

**Lemma 5.4.** *For any symmetric gus  $(X, M)$ , we have*

$$a_* \subseteq b_* \text{ \& } b <_X c \implies a \triangleleft_X c$$

for all  $a, b, c \in U_X$ .

**Theorem 5.5.** *For any symmetric gus  $(X, M)$ , we have  $\overline{\mathcal{S}_X} \cong \mathcal{U}(X)$ . That is, the localic completion of a symmetric gus is the point-free completion of the standard uniform formal topology induced by  $M$ . In particular, the localic completion of a symmetric gus is complete as a uniform formal topology.*

*Proof.* We define a binary relation  $r_X$  on  $U_X$  by

$$a r_X b \stackrel{\text{def}}{\iff} (\exists b' <_X b) a_* \subseteq b'_*,$$

and show that it is a surjective embedding from  $\overline{\mathcal{S}_X}$  to  $\mathcal{U}(X)$ .

- (1)  $r_X$  is a formal topology map: We check (FTM1), (FTM2), (FTM3a), and (FTM3b).

(FTM1) Let  $a = \mathbf{b}_d(x, \varepsilon) \in U_X$ . Then, we have  $a r_X \mathbf{b}_d(x, 2\varepsilon)$ , from which (FTM1) follows.

(FTM2) Let  $b, c \in U_X$ , and let  $a \in r_X^- b \downarrow r_X^- c$ . Then, there exist  $b' <_X b$  and  $c' <_X c$  such that  $a_* \subseteq b'_* \cap c'_*$ . Write  $b' = \mathbf{b}_{d_1}(x, \varepsilon)$  and  $c' = \mathbf{b}_{d_2}(y, \delta)$ ,

and choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\mathbf{b}_{d_1}(x, \varepsilon + 3\theta) <_X b$ , and  $\mathbf{b}_{d_2}(y, \delta + 3\theta) <_X c$ . Put  $\rho = \sup\{d_1, d_2\}$ , and let  $a' = \mathbf{b}_{\rho'}(z, \xi) \in \mathcal{C}_{\rho}^{\theta} \downarrow a$ . Then,  $a'_* \subseteq \mathbf{b}_{\rho}(z, 2\theta)_*$ . Since  $z \in a_*$ , we have  $d(x, z) + 3\theta < \varepsilon + 3\theta$ , so that  $\mathbf{b}_{\rho}(z, 3\theta) <_X \mathbf{b}_{d_1}(x, \varepsilon + 3\theta)$ . Similarly, we have  $\mathbf{b}_{\rho}(z, 3\theta) <_X \mathbf{b}_{d_2}(y, \delta + 3\theta)$ . Hence,  $a' r_X \mathbf{b}_{\rho}(z, 3\theta)$  and  $\mathbf{b}_{\rho}(z, 3\theta) \in b \downarrow c$ . Therefore,  $a \blacktriangleleft_X r_X^-(b \downarrow c)$  by (C2).

(FTM3a) By Lemma 3.10 (1).

(FTM3b) Preservation of the axiom (U1) follows from Lemma 3.10 (2). For (U2), let  $(d, \varepsilon) \in \text{Rad}(M)$ . Putting  $\delta = \varepsilon/2$ , we have  $U_X \blacktriangleleft_X \mathcal{C}_d^{\delta} \subseteq r_X^-\mathcal{C}_d^{\varepsilon}$  by (C2).

- (2)  $r_X$  is an embedding: We must show that  $a \blacktriangleleft_X r_X^- r_X^* \mathcal{A}_X \{a\}$  for all  $a \in U_X$ . Let  $a \in U_X$ , and let  $a' \in U_X$  such that  $a' \prec_X a$ . Then, there exists  $(d, \varepsilon) \in \text{Rad}(M)$  such that  $a' \prec_X^{(d, \varepsilon)} a$ . Choose  $\theta \in \mathbb{Q}^{>0}$  such that  $\theta < \varepsilon$ , and let  $b \in \mathcal{C}_d^{\theta} \downarrow a'$ . Then, there exists  $\mathbf{b}_d(x, \theta) \in \mathcal{C}_d^{\theta}$  such that  $b_* \subseteq \mathbf{b}_d(x, \theta)_*$ , and thus  $b r_X \mathbf{b}_d(x, \varepsilon)$ . Let  $b' \in r_X^- \mathbf{b}_d(x, \varepsilon)$ . Then, we have  $b'_* \subseteq \mathbf{b}_d(x, \varepsilon)_*$ . Since  $a'_* \not\subseteq \mathbf{b}_d(x, \varepsilon)_*$ , we have  $\mathbf{b}_d(x, \varepsilon)_* \subseteq a_*$ . Hence  $b' \preceq_X a$ , so  $\mathbf{b}_d(x, \varepsilon) \in r_X^* \mathcal{A}_X \{a\}$ . Therefore, by (C1) and (C2), we have  $a \blacktriangleleft_X r_X^- r_X^* \mathcal{A}_X \{a\}$ , as required.
- (3)  $r_X$  is a surjection: We must show that

$$r_X^- a \blacktriangleleft_X r_X^- U \implies a \triangleleft_X U$$

for all  $a \in U_X$  and  $U \subseteq U_X$ . Since  $b <_X a \implies b r_X a$  for all  $a, b \in U_X$ , it suffices to show that

$$a \blacktriangleleft_X r_X^- U \implies a \triangleleft_X U$$

for all  $a \in U_X$  and  $U \subseteq U_X$  by (U1). Given  $U \subseteq U_X$ , define a predicate  $\Phi_U$  on  $U_X$  by

$$\Phi_U(a) \stackrel{\text{def}}{\iff} (\forall b \in U_X) b_* \subseteq a_* \implies b \triangleleft_X U.$$

Then, it suffices to show that

$$a \blacktriangleleft_X r_X^- U \implies \Phi_U(a)$$

for all  $a \in U_X$ . This is proved by induction on  $\blacktriangleleft_X$ . We must check the conditions (ID1) – (ID3).

(ID1) Suppose that  $a \in r_X^- U$ , and let  $b \in U_X$  such that  $b_* \subseteq a_*$ . Then, there exist  $c \in U$  and  $c' <_X c$  such that  $a_* \subseteq c'_*$ . Thus  $b_* \subseteq c'_*$ , and hence,  $b \triangleleft_X c$  by Lemma 5.4. Therefore  $\Phi_U(a)$ .

(ID2) This directly follows from the definition of  $\preceq_X$ .

(ID3) We need to check the axioms (C1) and (C2').

(C1)  $\frac{(\forall b \prec_X a) \Phi_U(b)}{\Phi_U(a)}$ : Suppose that  $\Phi_U(b)$  holds for all  $b \prec_X a$ . Let  $c \in U_X$

such that  $c_* \subseteq a_*$ , and let  $b \in U_X$  such that  $b <_X c$ . Then,  $b \prec_X c$  by Lemma 5.3, and so  $b \prec_X a$ . Thus,  $\Phi_U(b)$ , and so  $b \triangleleft_X U$ . Hence  $c \triangleleft_X U$  by (U1). Therefore  $\Phi_U(a)$ .

(C2')  $\frac{(\forall b \in \mathcal{C}_d^{\varepsilon} \downarrow \preceq_X a) \Phi_U(b)}{\Phi_U(a)}$  for each  $(d, \varepsilon) \in \text{Rad}(M)$ : Let  $(d, \varepsilon) \in \text{Rad}(M)$ , and

suppose that  $\Phi_U(b)$  holds for all  $b \in \mathcal{C}_d^{\varepsilon} \downarrow \preceq_X a$ . Let  $c \in U_X$  such that  $c_* \subseteq a_*$ . Let  $b \in \mathcal{C}_d^{\varepsilon} \downarrow c$ . Then,  $b \in \mathcal{C}_d^{\varepsilon} \downarrow \preceq_X a$ , and so  $\Phi_U(b)$ . Thus,  $b \triangleleft_X U$ , and hence  $c \triangleleft_X U$  by (U2). Therefore  $\Phi_U(a)$ .  $\square$

## 6. FURTHER WORK

Curi [Cur06] developed a theory of uniform formal topologies that is different from that of Fox [Fox05]. In the same paper, Curi sketched another embedding from the category of uniform spaces<sup>7</sup> and uniformly continuous functions into uniform formal topologies.

Curi's embedding, however, sends a uniform space to a formal topology which has a usual topology induced by the uniformity. Hence, without Fan theorem, his embedding cannot be shown to preserve compactness and local compactness of uniform spaces as our localic completion does. This is one of our motivations to develop the localic completion of generalised uniform spaces. However, a uniform formal topology that arises from Curi's embedding might be 'completed' in a suitable sense to give a topology equivalent to the localic completion. Development of completions of uniform formal topologies in Curi's sense and its comparison to our localic completion and Fox's notion of uniform formal topology are left for the future.

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<sup>7</sup>Curi's notion of uniform space is that of symmetric generalised uniform space in our terminology.

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